

Relatively hyperbolic groups: geometry and quasi-isometric invariance

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Abstract

In this paper it is proved that relative hyperbolicity is an invariant of quasi-isometry. As a byproduct of the arguments, simplified definitions of relative hyperbolicity are obtained. In particular we obtain a new definition very similar to the one of hyperbolicity, relying on the existence for every quasi-geodesic triangle of a central left coset of peripheral subgroup.

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1 Introduction

1.1 Rigidity result

M. Gromov asked ([Gro87],[Gro93]) what properties of infinite finitely generated groups are invariant by quasi-isometry. Such properties are sometimes called *geometric*, while a class of groups defined by a geometric property is called *rigid*.

Examples of rigid/non-rigid classes of groups:

1. the class of virtually nilpotent groups is rigid [Gro81];
2. the class of virtually solvable groups is not rigid [Dyu00]; but smaller classes of virtually solvable groups are rigid ([FM98], [FM99], [EFW05]);
3. amenability is a geometric property;
4. property (T) is not geometric (see for instance [Val04]);
5. hyperbolicity is a geometric property [Gro87];
6. different classes of lattices of semisimple groups are rigid (this statement includes many deep results of different authors; see [Far97] and [Dru04] for surveys of these results).

Recall that a group is said to *virtually* satisfy a property (P) if a finite index subgroup of it has property (P).

The present paper focuses on the class of relatively hyperbolic groups.¹ This notion was introduced by M. Gromov in [Gro87]. Other definitions, as well as developments of the theory of relatively hyperbolic groups can be found in [Bow97], [Far98], [Dah03b], [Yam04], [DS05b], [Osi06]. In § 1.2 and § 1.3 we discuss in more detail different ways to define relative hyperbolicity.

Examples of relatively hyperbolic groups:

1. a hyperbolic group is hyperbolic relative to $\{1\}$;
2. an amalgamated product $A *_F B$, where F is finite, is hyperbolic relative to A and B ; more generally, fundamental groups of finite graphs of groups with finite edge groups are hyperbolic relative to the vertex groups [Bow97];
3. fundamental groups of complete finite volume manifolds of pinched negative sectional curvature are hyperbolic relative to the fundamental groups of their cusps ([Bow97], [Far98]);
4. fundamental groups of (non-geometric) Haken manifolds with at least one hyperbolic component are hyperbolic relative to fundamental groups of maximal graph-manifold components and to fundamental groups of tori and Klein bottles not contained in a graph-manifold component;
5. fully residually free groups, also known as limit groups, are hyperbolic relative to their maximal Abelian non-cyclic subgroups [Dah03a]. Moreover they are CAT(0) with isolated flats [AB05].

Note that there are also some interesting examples of groups displaying a sort of “intermediate” relative hyperbolicity: they are weakly relatively hyperbolic, not (strongly) relatively hyperbolic, but nevertheless they have some common features with (strongly) relatively hyperbolic groups, for instance their asymptotic cones have a similar metric structure. Such groups are the mapping class groups of surfaces of complexity at least two ([Beh05], [BDM05]), fundamental groups of 3-dimensional graph manifolds ([KL98], [KKL98], [BDM05]), as well as many Artin groups ([KS04], [BDM05]).

Recently, relatively hyperbolic groups have been used to construct examples of infinite finitely generated groups with unusual properties. Thus in [Osi04] it is proved that there exist torsion-free two-generated groups with exactly two conjugacy classes.

¹By relatively hyperbolic group we mean what is sometimes called in the literature *strongly relatively hyperbolic group*, in contrast with *weakly relatively hyperbolic group*.

Convention 1.1. Throughout the paper all relatively hyperbolic groups are assumed to be *finitely generated* and hyperbolic relative to *finitely many proper subgroups of finite type*.

We also use the following terminology: if a group G is hyperbolic relative to subgroups H_1, \dots, H_m then the subgroups H_1, \dots, H_m are called *peripheral subgroups*.

The present paper gives an affirmative answer to the question whether relative hyperbolicity is a quasi-isometry invariant (formulated also as Problem 1.15 in [DS05b]).

Theorem 1.2 (relative hyperbolicity is geometric, Theorem 5.12). *Let G be a group hyperbolic relative to a family of subgroups H_1, \dots, H_n . If a group G' is quasi-isometric to G then G' is hyperbolic relative to H'_1, \dots, H'_m , where each H'_i can be embedded quasi-isometrically in H_j for some $j = j(i) \in \{1, 2, \dots, n\}$.*

Rigidity has previously been proved for some sub-classes of relatively hyperbolic groups (with stronger versions of rigidity theorems): non-uniform lattices in rank one semisimple groups different from $\mathrm{SL}(2, \mathbb{R})$ [Sch96], fundamental groups of non-geometric Haken manifolds with at least one hyperbolic component ([KL95], [KL97]), fundamental groups of graphs of groups with finite edge groups [PW02].

In the full generality assumed in Theorem 1.2, the stronger statement that each subgroup H'_i is quasi-isometric to some subgroup H_j cannot hold. This can be seen in the example when $G = G' = A * B * C * D$, with G hyperbolic relative to $\{A * B, C * D\}$ and G' hyperbolic relative to $\{A, B, C, D\}$. In [BDM05] it is shown that if in Theorem 1.2 it is moreover assumed that each peripheral subgroup H_i is *not relatively hyperbolic* then the rigidity result holds, moreover each H'_i is quasi-isometric to some H_j . This generalizes previous results from [DS05b]. The proof of Theorem 1.2 is completely different from the proofs in [DS05b] and in [BDM05]. The main ingredient in [BDM05] is the following result, proved using results from the present paper: given a group G hyperbolic relative to H_1, \dots, H_n , every quasi-isometric embedding into G of a group which is not relatively hyperbolic has its image in a bounded radius tubular neighborhood of a left coset gH_i ; moreover the radius of the neighborhood depends only on G, H_1, \dots, H_n and on the constants of quasi-isometry, *not* on the domain of the quasi-isometry [BDM05, Theorem 4.1].

The main steps in the proof of Theorem 1.2 are explained in what follows.

1.2 Metric and algebraic relative hyperbolicity

In order to study rigidity it is necessary to have a definition of relative hyperbolicity of a group only in terms of its Cayley graphs. Most definitions (except the ones in [DS05b] and in [Osi06]) use not only a Cayley graph of the group but also a metric space obtained from this graph by gluing to each left coset of a peripheral subgroup some geometric object (a hyperbolic horoball [Gro87], countably many edges with one common endpoint [Far98] etc).

In what follows, we recall definitions provided in [DS05b].

A complete geodesic metric space \mathbb{F} is *tree-graded with respect to a collection \mathcal{P}* of closed geodesic subsets (called *pieces*), if the following two properties are satisfied:

(T_1) two different pieces have at most one point in common;

(T_2) any simple non-trivial geodesic triangle is contained in one piece.

A similar, though not equivalent, notion has been introduced in [KKL98] under the name of space of type I.

A metric space X is *asymptotically tree-graded with respect to a collection of subsets \mathcal{A}* if every asymptotic cone of X is tree-graded with respect to the collection of limit sets of sequences in \mathcal{A} . A definition of asymptotic cones of metric spaces, and of limit sets can be found in Section 2.2.

Equivalently, X is asymptotically tree-graded with respect to \mathcal{A} if the following three geometric properties are satisfied (for details see Theorem 4.1 in [DS05b] or Theorem 4.8 in this paper):

- (α_1) finite radius tubular neighborhoods of distinct elements in \mathcal{A} are either disjoint or intersect in sets of uniformly bounded diameter;
- (α_2) a geodesic with endpoints at distance at most one third of its length from a set A in \mathcal{A} intersects a tubular neighborhood of A of uniformly bounded radius;
- (α_3) any fat geodesic polygon is contained in a tubular neighborhood of a set A in \mathcal{A} of uniformly bounded radius (here the meaning of “fat” is the contrary of “thin” in its metric hyperbolic sense; see Definition 4.5).

The space X is *properly* asymptotically tree-graded with respect to \mathcal{A} if it is not contained in any finite radius tubular neighborhood of a subset in \mathcal{A} .

Convention 1.3. In what follows we assume that all asymptotically tree-graded metric spaces are properly asymptotically tree-graded.

The notion of asymptotically tree-graded metric space is a metric version for the relative hyperbolicity of groups. Other similar notions can be found in [BF01], and in [HK05] in the context of CAT(0) metric spaces. The fact that the metric definition is coherent with the definition for groups is illustrated by the following result.

Theorem 1.4 ([DS05b], Theorem 1.11 and Appendix). *A finitely generated group G is hyperbolic relative to H_1, \dots, H_m if and only if G is asymptotically tree-graded with respect to the collection of left cosets $\mathcal{L} = \{gH_i ; g \in G/H_i, i \in \{1, 2, \dots, m\}\}$.*

The equivalence in Theorem 1.4 suggests the following question, which appears as Problem 1.16 in [DS05b]: if a group is asymptotically tree-graded in a metric sense, that is with respect to a collection of subsets \mathcal{A} , does it follow that it is relatively hyperbolic with respect to some finite family of subgroups? The implication was previously known to be true only under some restrictive metric conditions on \mathcal{A} (see [DS05b, Theorem 5.13] and [BDM05]).

We answer this question in the affirmative.

Theorem 1.5 (Theorem 5.1). *Let G be an infinite finitely generated group asymptotically tree-graded with respect to a collection of subsets \mathcal{A} . Then G is relatively hyperbolic with respect to some subgroups H_1, \dots, H_m , such that every H_i is contained in a bounded radius tubular neighborhood of a set $A_i \in \mathcal{A}$.*

Theorem 1.5 implies Theorem 1.2. Indeed, a group quasi-isometric to a relatively hyperbolic group is asymptotically tree-graded as a metric space with respect to the images by quasi-isometry of the left cosets of peripheral subgroups [DS05b, Theorem 5.1].

An outline of the proof of Theorem 1.5 will be given in the following sections.

Theorem 1.5 is optimal in the sense that if the group G and the collection \mathcal{A} satisfy less properties than those required for asymptotically tree-graded metric spaces then the group G

may not be relatively hyperbolic. This is shown by the examples of groups constructed in [BDM05, §7.1] and in [OOS06]. These groups are not relatively hyperbolic, although they contain a collection of subsets \mathcal{A} such that all the asymptotic cones are tree-graded with respect to *some* limits of sequences in \mathcal{A} . But in each cone, not all the limits of sequences in \mathcal{A} are considered as pieces: there are limits which are geodesic lines, and different such lines intersect in more than one point. The subsets in \mathcal{A} do not satisfy property (α_1) .

1.3 New definitions of relative hyperbolicity

If a group has an asymptotically tree-graded structure equivariant with respect to left translations, then a standard argument shows that the group is relatively hyperbolic (see Proposition 5.5). Thus, the main step in the proof of Theorem 1.5 is to construct an equivariant asymptotically tree-graded structure on a group out of an arbitrary asymptotically tree-graded structure. A natural idea is to consider all the translated asymptotically tree-graded structures $g\mathcal{A} = \{gA ; A \in \mathcal{A}\}$ of a given asymptotically tree-graded structure \mathcal{A} on a group G , and to take non-empty intersections of the form $\bigcap_{g \in G} gA_g$, with $A_g \in \mathcal{A}$. To make such an argument work, it is necessary that the asymptotically tree-graded properties behave well with respect to intersections. The following modification of the list of three geometric properties defining an asymptotically tree-graded metric space ensures this good behavior with respect to intersections.

Theorem 1.6 (Theorem 4.21). *Let (X, dist) be a geodesic metric space and let \mathcal{A} be a collection of subsets of X . The space X is asymptotically tree-graded with respect to \mathcal{A} if and only if property (α_1) and the following two properties are satisfied:*

- (β_2) a geodesic with endpoints at distance at most $\frac{1}{k}$ of its length from a set A in \mathcal{A} (with k large enough) has its middle third contained in a tubular neighborhood of A of uniformly bounded radius;*
- (β_3) any fat geodesic hexagon is contained in a tubular neighborhood of a set A in \mathcal{A} of uniformly bounded radius.*

It is not difficult to replace property (α_2) by (β_2) , using results in [DS05b]. But replacing (α_3) by (β_3) requires extra work. Property (β_3) implies that (T_2) holds in any asymptotic cone for simple triangles whose edges are limits of sequences of geodesics (Proposition 4.13). But generically a geodesic in an asymptotic cone of a group is not limit of a sequence of geodesics (see the example in the end of § 2.2). In order to ensure (T_2) for an arbitrary geodesic triangle the argument in [DS05b] was to prove that such a triangle can be approximated by a geodesic triangle which is limit of a sequence of fat polygons with the same number m of edges (see Lemma 4.14 in this paper). The number m of edges must increase when the constant of approximation decreases. This approximation result and property (α_3) imply (T_2) . In this paper we show (Corollary 4.19) that if property (T_1) holds in every asymptotic cone, an inductive argument allows to deduce (T_2) from (β_3) .

Asymptotically tree-graded metric spaces have a property that strongly reminds of hyperbolic metric spaces. A metric space is hyperbolic if and only if the edges of every quasi-geodesic triangle intersect a ball of uniformly bounded radius [Gro87, §6]. A space X that is asymptotically tree-graded with respect to a collection of subsets \mathcal{A} has the following property [DS05b]:

- (*) the edges of any quasi-geodesic triangle in X either intersect a finite radius ball or a finite radius tubular neighborhood of a subset in \mathcal{A} . Moreover, in the latter case the distance between the entrance points into the tubular neighborhood of two edges with common origin is uniformly bounded.*

If (X, \mathcal{A}) satisfy property $(*)$ then the space X is called $(*)$ -asymptotically tree-graded with respect to \mathcal{A} [DS05a]. This notion is weaker than the notion of asymptotically tree-graded metric space (see Remark 4.33, (2)). Property $(*)$ was essential in the proof of the fact that the property of Rapid Decay transfers from the peripheral subgroups H_1, \dots, H_m of a relatively hyperbolic group to the group itself [DS05a]. A version of property $(*)$ in the context of CAT(0) spaces appears in [Hru04], where it is called the Relatively Thin Triangle Property.

A natural question to ask is under what additional conditions is a $(*)$ -asymptotically tree-graded metric space also asymptotically tree-graded. The arguments used to prove Theorem 1.6 can be adapted to answer this question.

Theorem 1.7 (Theorem 4.34). *Let (X, dist) and \mathcal{A} be as in Theorem 1.6. The metric space X is asymptotically tree-graded with respect to \mathcal{A} if and only if (X, \mathcal{A}) satisfy properties (α_1) and (α_2) , and moreover X is $(*)$ -asymptotically tree-graded with respect to \mathcal{A} .*

1.4 Organization of the paper

Section 2 contains preliminaries on asymptotic cones, as well as notation used throughout the paper.

In Section 3 are recalled some basic facts about tree-graded spaces. Proposition 3.9 proved in the same section is very useful in different arguments deducing the general property (T_2) from (T_1) , and (T_2) restricted to some particular cases.

Section 4 begins with a short overview of properties of asymptotically tree-graded metric spaces. In §4.2 an induction argument and Proposition 3.9 are used to show the following central result. Denote by (Π_3) the property (T_2) restricted to triangles with edges limits of sequences of geodesics. If in an asymptotic cone $\text{Con}_\omega(X)$ of a metric space X a collection \mathcal{A}_ω of closed subsets satisfies (T_1) and (Π_3) then \mathcal{A}_ω satisfies (T_2) in full generality (Corollary 4.18).

This statement is the main ingredient in the proof of Theorem 1.6, given in §4.3. It also plays a central part in the proof of Theorem 1.7 given in §4.4. Another difficult step in the proof of Theorem 1.7 is to deduce from properties $(*)$, (α_1) and (α_2) the fact that fat quadrilaterals are contained in finite radius tubular neighborhoods of subsets in \mathcal{A} (Lemma 4.38). Once this last statement proved, from it as well as from property $(*)$ and Proposition 3.6 can be deduced property (Π_3) . Corollary 4.18 allows to finish the argument.

Theorem 1.5 is proved in Section 5. The first and most difficult step of the proof is to construct from a given asymptotically tree-graded structure on a group an equivariant asymptotically tree-graded structure. The subsets in the new asymptotically tree-graded structure are indexed by equivalence classes of fat hexagons. A simple argument then shows that the existence of an equivariant asymptotically tree-graded structure implies that the group is relatively hyperbolic (Proposition 5.5). This completes the proof of Theorem 1.5 and thus of Theorem 1.2.

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2 Preliminaries

2.1 Definitions and notation

Let Y be a subset in a metric space (X, dist) . We denote by $\mathcal{N}_\delta(Y)$ the set $\{x \in X \mid \text{dist}(x, Y) < \delta\}$, which we call the δ -tubular neighborhood of Y . We denote by $\overline{\mathcal{N}}_\delta(Y)$ the set $\{x \in X \mid \text{dist}(x, Y) \leq \delta\}$, called the δ -closed tubular neighborhood of Y .

When Y is a singleton y , we also use the notation $B(y, \delta)$ and respectively $\overline{B}(y, \delta)$.

Definition 2.1. An action of a group G on a metric space X is called \mathcal{K} -transitive, where \mathcal{K} is a non-negative constant, if for every $x \in X$ the closed tubular neighborhood $\overline{\mathcal{N}}_{\mathcal{K}}(Gx)$ of the orbit of x coincides with X .

An (L, C) -quasi-isometric embedding of a metric space (X, dist_X) into a metric space (Y, dist_Y) is a map $\mathfrak{q} : X \rightarrow Y$ such that for every $x_1, x_2 \in X$,

$$\frac{1}{L} \text{dist}_X(x_1, x_2) - C \leq \text{dist}_Y(\mathfrak{q}(x_1), \mathfrak{q}(x_2)) \leq L \text{dist}_X(x_1, x_2) + C, \quad (1)$$

for some constants $L \geq 1$ and $C \geq 0$.

If moreover Y is contained in the C -tubular neighborhood of $\mathfrak{q}(X)$ then \mathfrak{q} is called an (L, C) -quasi-isometry. In this case there exists an (L, C) -quasi-isometry $\bar{\mathfrak{q}} : Y \rightarrow X$ such that $\bar{\mathfrak{q}} \circ \mathfrak{q}$ and $\mathfrak{q} \circ \bar{\mathfrak{q}}$ are at uniformly bounded distance from the respective identity maps [GdlH90]. The quasi-isometry $\bar{\mathfrak{q}}$ is called *quasi-converse* of \mathfrak{q} .

If $\mathfrak{q} : [a, b] \rightarrow X$ is an (L, C) -quasi-isometric embedding then \mathfrak{q} is called an (L, C) -quasi-geodesic (segment) in X . The same name is used for the image of \mathfrak{q} .

Notation 2.2. For every quasi-geodesic segment \mathfrak{q} in a metric space X , we denote the origin of \mathfrak{q} by \mathfrak{q}_- and the endpoint of \mathfrak{q} by \mathfrak{q}_+ .

If $\mathfrak{q}_i : [0, \ell_i] \rightarrow X$, $i = 1, 2$, are two quasi-geodesic segments with $\mathfrak{q}_1(\ell_1) = \mathfrak{q}_2(0)$, then we denote by $\mathfrak{q}_1 \sqcup \mathfrak{q}_2$ the map $\mathfrak{q} : [0, \ell_1 + \ell_2] \rightarrow X$ defined by $\mathfrak{q}(t) = \mathfrak{q}_1(t)$ for $t \in [0, \ell_1]$ and $\mathfrak{q}(t) = \mathfrak{q}_2(t - \ell_1)$ for $t \in [\ell_1, \ell_1 + \ell_2]$.

If an (L, C) -quasi-geodesic \mathfrak{q} is L -Lipschitz then \mathfrak{q} is called an (L, C) -almost geodesic.

2.2 Asymptotic cones of a metric space

The notion of asymptotic cone of a metric space was used implicitly in [Gro81], and it was defined in full generality and studied in [dDW84] and [Gro93]. For the definition, one needs the notion of *non-principal ultrafilter*. This is a finitely additive measure ω defined on the set of all subsets of \mathbb{N} (or, more generally, of a countable set) and taking values in $\{0, 1\}$, such that $\omega(F) = 0$ for every finite subset F of \mathbb{N} .

Convention 2.3. Throughout the paper all ultrafilters are non-principal, therefore we will omit mentioning it each time.

Notation 2.4. Let A_n and B_n be two sequences of objects and let \mathcal{R} be a relation that can be established between A_n and B_n for every $n \in \mathbb{N}$. We write $A_n \mathcal{R}_\omega B_n$ if and only if $A_n \mathcal{R} B_n$ ω -almost surely, that is

$$\omega(\{n \in \mathbb{N} \mid A_n \mathcal{R} B_n\}) = 1.$$

Examples: $=_\omega$, $<_\omega$, \subset_ω .

Given an ultrafilter ω , an ω -limit $\lim_\omega x_n$ of a sequence (x_n) in a topological space X is an element $x \in X$ such that for every neighborhood \mathcal{N} of x , $x_n \in_\omega \mathcal{N}$. In a Hausdorff separable space if the ω -limit of a sequence exists then it is unique. If (x_n) is contained in a compact space then it has an ω -limit [Bou65].

Given a space X one can define its *ultrapower* X^ω as the quotient $X^\mathbb{N} / \approx$, where $(x_n) \approx (y_n)$ if $x_n =_\omega y_n$.

Let now (X, dist) be a metric space, e a fixed element in its ultrapower X^ω , (e_n) a representative of e , and $d = (d_n)$ a sequence of numbers in $(0, +\infty)$ such that $\lim_\omega d_n = +\infty$.

Consider

$$\mathcal{S}_e(X) = \left\{ (x_n) \in X^\mathbb{N}; \text{ there exists } M_x \text{ such that } \text{dist}(x_n, e_n) \leq_\omega M_x d_n \right\}. \quad (2)$$

Define the equivalence relation

$$(x_n) \sim (y_n) \Leftrightarrow \lim_\omega \frac{\text{dist}(x_n, y_n)}{d_n} = 0.$$

The quotient space $\mathcal{S}_e(X)/\sim$ is denoted by $\text{Con}_\omega(X; e, d)$ and it is called *the asymptotic cone of X with respect to the ultrafilter ω , the scaling sequence d and the sequence of observation centers e* . It is endowed with the natural metric dist_ω defined by

$$\text{dist}_\omega(x, y) = \lim_\omega \frac{\text{dist}(x_n, y_n)}{d_n}.$$

Every asymptotic cone is a complete metric space.

A sequence of subsets (A_n) in X gives rise to a *limit subset* in the cone, defined by

$$\lim_\omega (A_n) = \left\{ \lim_\omega (a_n) \mid a_n \in_\omega A_n \right\}.$$

If $\lim_\omega \frac{\text{dist}(e_n, A_n)}{d_n} = +\infty$ then $\lim_\omega (A_n) = \emptyset$. Every non-empty limit subset $\lim_\omega (A_n)$ is closed.

If each set A_n is a geodesic \mathfrak{g}_n with length of order $O(d_n)$ and $\lim_\omega (\mathfrak{g}_n)$ is non-empty, then it is a geodesic in $\text{Con}_\omega(X; e, d)$. Therefore if X is a geodesic space then every asymptotic cone of it is geodesic.

Definition 2.5. We call a geodesic in $\text{Con}_\omega(X; e, d)$ which appears as $\lim_\omega (\mathfrak{g}_n)$ with \mathfrak{g}_n geodesics in X a *limit geodesic*.

Not every geodesic in $\text{Con}_\omega(X; e, d)$ is a limit geodesic, not even in the particular case when X is a group of finite type with a word metric.

Example of group with continuously many non-limit geodesics in an asymptotic cone:

On the two-dimensional unit sphere \mathbb{S}^2 consider a family of horizontal circles, and a family of vertical circles in parallel planes, such that two consecutive circles in each family are at spherical distance $\frac{\pi}{2^k}$, and such that the North and the South points are on one vertical circle, and are at distance $\frac{\pi}{2^k}$ from two respective horizontal circles.

The two families of circles compose a spherical grid Γ'_k . We have that $\Gamma'_k \subset \Gamma'_{k+1}$. Let Γ''_k be the graph obtained from Γ'_k by joining with spherical geodesics all pairs of vertices not on the same vertical or horizontal circle, and at distance at most $\frac{\pi}{\sqrt{2}^k}$. Let Γ_k be the graph obtained from Γ''_k by deleting all the vertical edges of length $\frac{\pi}{2^k}$ above the Equator, except the one having the East point $(1, 0, 0)$ as an endpoint, and replacing each of them by a path of double length $\frac{\pi}{2^{k-1}}$. Let dist_k be the shortest-path metric on Γ_k .

Proposition 7.26 from [DS05b] applied to the sequence of graphs $(\Gamma_k, \text{dist}_k)$, and Lemma 7.5 from the same paper imply that there exists a two-generated and recursively presented group G with one asymptotic cone tree-graded, with all pieces isometric to \mathbb{S}^2 . Moreover, from the construction of G it follows that in each of the pieces, for an appropriate choice of the North,

South and East points, all geodesics joining North and South and not containing East are not limit geodesics.

The same argument as in [DS05b, §7] allows in fact to construct a two-generated and recursively presented group with continuously many non-homeomorphic asymptotic cones with the property that continuously many geodesics in each of them are not limit geodesics.

3 Tree-graded metric spaces

3.1 Definition and properties

The notion of tree-graded metric space has been introduced in [DS05b]. In this paper we use the following version of this notion. Recall that a subset A in a geodesic metric space X is called *geodesic* if every two points in A can be joined by a geodesic contained in A .

Definition 3.1. Let \mathbb{F} be a complete geodesic metric space and let \mathcal{P} be a collection of closed geodesic subsets, called *pieces*. Suppose that the following two properties are satisfied:

- (T_1) Every two different pieces have at most one point in common.
- (T_2) Every simple non-trivial geodesic triangle in \mathbb{F} is contained in one piece.

Then we say that the space \mathbb{F} is *tree-graded with respect to \mathcal{P}* .

When there is no risk of confusion as to the set \mathcal{P} , we simply say that \mathbb{F} is *tree-graded*.

Remarks 3.2 (pieces need not cover the space). (1) In [DS05b] trivial geodesic triangles are allowed in property (T_2). This is equivalent to asking that \mathbb{F} is covered by the pieces in \mathcal{P} . In the present paper we remove this convention. The reason is that a main purpose when introducing the notion of tree-graded space is to produce a convenient notion of relatively hyperbolic metric space (called asymptotically tree-graded metric space in [DS05b] and in this paper, see Definition 4.1). The condition that pieces cover \mathbb{F} produces some unnatural restrictions for a space to be asymptotically tree-graded (i.e. relatively hyperbolic) with respect to a list of subsets. See Remark 4.11 for details.

- (2) Possibly \mathcal{P} is empty, in which case \mathbb{F} is a real tree.
- (3) When a group G acts transitively on \mathbb{F} (for instance when \mathbb{F} is an asymptotic cone of a group) and G permutes the pieces, the condition that pieces cover \mathbb{F} is automatically satisfied.

All properties of tree-graded spaces in [DS05b, §2.1] hold with the new definition 3.1, as none of the proofs uses the property that pieces cover the space. In particular one has the following results.

Lemma 3.3 ([DS05b], §2.1). *Let x be an arbitrary point in \mathbb{F} and let T_x be the set of points $y \in \mathbb{F}$ which can be joined to x by a topological arc intersecting every piece in at most one point.*

The subset T_x is a real tree and a closed subset of \mathbb{F} , and every topological arc joining two points in T_x is contained in T_x . Moreover, for every $y \in T_x$, $T_y = T_x$.

Definition 3.4. A subset T_x as in Lemma 3.3 is called a *transversal tree* in \mathbb{F} .

In [KKL98] is defined the notion of space of type I, which is equivalent to that of a tree-graded space with the extra property that for every x the transversal tree T_x is a geodesically complete tree which branches everywhere.

Remark 3.5. One can ensure that pieces in a tree-graded space cover it by adding to the list of pieces the transversal trees. Thus a tree-graded space \mathbb{F} with set of pieces \mathcal{P} in the sense of Definition 3.1 can be seen as tree-graded in the sense of Definition 2.1 in [DS05b] with respect to a set of pieces \mathcal{P}' such that $\mathcal{P}' \setminus \mathcal{P}$ is a collection of real trees.

3.2 Topological bigons contained in pieces

Definition 3.6. We call *topological bigon* (\mathcal{T} -bigon, in short) formed by two topological arcs \mathbf{g}_1 and \mathbf{g}_2 a union of a sub-arc \mathbf{g}'_1 of \mathbf{g}_1 with a sub-arc \mathbf{g}'_2 of \mathbf{g}_2 such that \mathbf{g}'_1 and \mathbf{g}'_2 have common endpoints x and y . The *endpoints of the \mathcal{T} -bigon* are the points x and y . The *interior of the \mathcal{T} -bigon* is the set $\mathbf{g}'_1 \cup \mathbf{g}'_2 \setminus \{x, y\}$.

If \mathbf{g}'_1 and \mathbf{g}'_2 intersect only in their endpoints then the \mathcal{T} -bigon is called *simple* (in fact it is a simple loop in this case).

Note that a \mathcal{T} -bigon with non-empty interior cannot be trivial, i.e. reduced to a point.

The results in this section are useful in arguments aiming to prove property (T_2) for a collection of closed subsets of a metric space. In several contexts it proves necessary to deduce from (T_1) , and (T_2) satisfied only for some special type of geodesic bigons, the general property (T_2) .

Lemma 3.7. *Let \mathbf{g}_1 and \mathbf{g}_2 be two topological arcs with common endpoints. Then every point $z \in \mathbf{g}_1 \setminus \mathbf{g}_2$ is in the interior of a simple \mathcal{T} -bigon formed by \mathbf{g}_1 and \mathbf{g}_2 .*

Proof. For $i = 1, 2$, $\mathbf{g}_i : [0, \ell_i] \rightarrow Y$ is a topological embedding. Let $t \in [0, \ell_1]$ be such that $\mathbf{g}_1(t) = z$. The set $K = \mathbf{g}_1^{-1}(\mathbf{g}_2([0, \ell_2]))$ is a compact set not containing t . Let r be the maximal element of the compact set $K \cap [0, t]$, and let s be the minimal element of the compact set $K \cap [t, \ell_1]$. Then $\mathbf{g}_1(r) = \mathbf{g}_2(r')$ and $\mathbf{g}_1(s) = \mathbf{g}_2(s')$ for some $r', s' \in [0, \ell_2]$. The union of \mathbf{g}_1 restricted to $[r, s]$ with \mathbf{g}_2 restricted to $[r', s']$ is a simple \mathcal{T} -bigon formed by \mathbf{g}_1 and \mathbf{g}_2 , containing z in its interior. \square

Lemma 3.8. *Let Y be a metric space and let \mathcal{B} be a collection of subsets of Y , \mathcal{B} satisfying property (T_1) .*

Let \mathbf{g}_1 and \mathbf{g}_2 be two topological arcs with common endpoints and with the property that any non-trivial simple \mathcal{T} -bigon formed by \mathbf{g}_1 and \mathbf{g}_2 is contained in a subset in \mathcal{B} .

If \mathbf{g}_1 is contained in $B \in \mathcal{B}$ then \mathbf{g}_2 is contained in B .

Proof. Take z an arbitrary point in $\mathbf{g}_2 \setminus \mathbf{g}_1$. By Lemma 3.7 the point z is in the interior of a simple \mathcal{T} -bigon formed by \mathbf{g}_1 and \mathbf{g}_2 , of endpoints z_1, z_2 . By hypothesis this \mathcal{T} -bigon is contained in a subset $B_z \in \mathcal{B}$. As $\{z_1, z_2\}$ is in $B \cap B_z$ it follows by (T_1) that $B_z = B$ and that $z \in B$. \square

Proposition 3.9. *Let Y be a metric space and let \mathcal{B} be a collection of closed subsets of Y , \mathcal{B} with property (T_1) .*

Let \mathcal{L}_1 and \mathbf{g}_1 be two topological arcs with common endpoints u, v . Let \mathcal{L}_2 and \mathbf{g}_2 be two, possibly identical, topological arcs with common endpoints v, w . Assume that:

- (1) $\mathcal{L}_1 \cap \mathcal{L}_2 = \{v\}$;
- (2) $\mathbf{g}_1 \cap \mathbf{g}_2$ contains a point $a \neq v$;
- (3) all non-trivial simple \mathcal{T} -bigons formed either by \mathbf{g}_1 and \mathbf{g}_2 , or by \mathbf{g}_i and \mathcal{L}_i , $i = 1, 2$, are contained in a subset in \mathcal{B} .

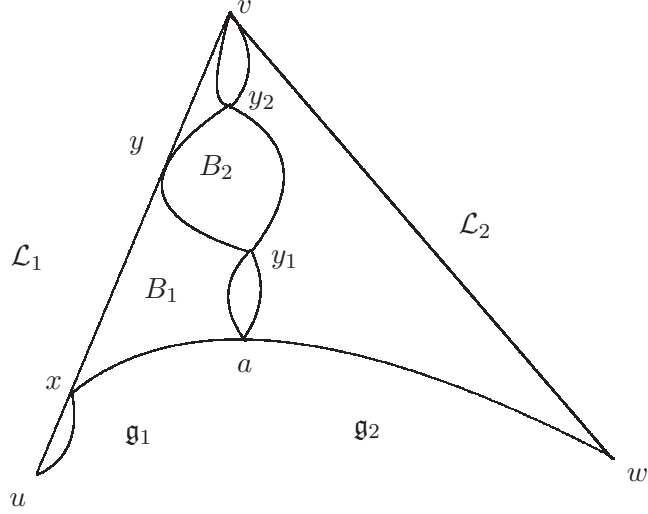


Figure 1: Step 1.

Then the \mathcal{T} -bigon formed by \mathbf{g}_1 and \mathbf{g}_2 with endpoints a and v is contained in a subset in \mathcal{B} .

Proof. STEP 1. Let \mathbf{g}'_i denote the sub-arc of \mathbf{g}_i of endpoints a and v , $i = 1, 2$.

We prove that there exists $b \in \mathbf{g}'_1 \cap \mathbf{g}'_2 \setminus \{a\}$, such that the \mathcal{T} -bigon formed by \mathbf{g}'_1 and \mathbf{g}'_2 of endpoints a, b is contained in some $B \in \mathcal{B}$.

Hypothesis (1) implies that either $a \notin \mathcal{L}_1$ or $a \notin \mathcal{L}_2$. Without loss of generality we may assume that $a \notin \mathcal{L}_1$. Then a is in the interior of a simple \mathcal{T} -bigon formed by \mathcal{L}_1 and \mathbf{g}_1 , of endpoints x and y , with y on \mathbf{g}'_1 . Property (3) implies that this \mathcal{T} -bigon is contained in a set $B_1 \in \mathcal{B}$.

If $y \in \mathbf{g}'_2$ then take $b = y$.

Assume that $y \notin \mathbf{g}'_2$. Then y is in the interior of a simple \mathcal{T} -bigon formed by \mathbf{g}'_1 and \mathbf{g}'_2 , of endpoints y_1, y_2 (with y_2 closer to v than y_1 on \mathbf{g}'_1). By (3) this \mathcal{T} -bigon is contained in some $B_2 \in \mathcal{B}$. The intersection $B_1 \cap B_2$ contains $\{y, y_1\}$ hence by (T_1) we have that $B_1 = B_2 = B$. Take $b = y_2$.

The sub-arc of \mathbf{g}'_1 with endpoints a and b is contained in B . By property (3) we can apply Lemma 3.8 and obtain that the sub-arc of \mathbf{g}'_2 in between a and b is also contained in B .

STEP 2. Let \mathcal{E} be the set of points $b \in \mathbf{g}'_1 \cap \mathbf{g}'_2 \setminus \{a\}$, such that the \mathcal{T} -bigon formed by \mathbf{g}'_1 and \mathbf{g}'_2 of endpoints a, b is contained in some $B \in \mathcal{B}$. We prove that there exists $c \in \mathcal{E}$ such that \mathbf{g}_1 between c and v contains no other point from \mathcal{E} .

Note that by property (T_1) of \mathcal{B} all \mathcal{T} -bigons of endpoints a and b , for some $b \in \mathcal{E}$, are contained in the same $B_0 \in \mathcal{B}$.

Let $\varphi : [0, \ell] \rightarrow Y$ be a parametrization of \mathbf{g}_1 , $\varphi(\ell) = v$, and let r be $\varphi^{-1}(a)$. The pre-image $\mathcal{E}' = \varphi^{-1}(\mathcal{E})$ is contained in $(r, \ell]$. Let T be the supremum of \mathcal{E}' . Then $T = \lim t_n$ for some increasing sequence (t_n) in \mathcal{E}' , hence $c = \varphi(T)$ is the limit of the sequence of points $b_n = \varphi(t_n) \in \mathcal{E}$. Obviously $c \in \mathbf{g}'_1 \cap \mathbf{g}'_2 \setminus \{a\}$. Since $B_0 \in \mathcal{B}$ is closed and $b_n \in B_0$, it follows that

$c \in B_0$. Thus the sub-arc of \mathfrak{g}'_1 between a and c is completely contained in B_0 . By Lemma 3.8 and property (3), the \mathcal{T} -bigon formed by \mathfrak{g}'_1 and \mathfrak{g}'_2 of endpoints a and c is in B_0 .

STEP 3. We prove that the point c obtained in Step 2 coincides with v .

Assume that $c \neq v$. Step 1 applied to the point c instead of a implies that there exists $d \in \mathfrak{g}'_1 \cap \mathfrak{g}'_2 \setminus \{c\}$, d between c and v on both \mathfrak{g}'_1 and \mathfrak{g}'_2 , such that the \mathcal{T} -bigon formed by \mathfrak{g}'_1 and \mathfrak{g}'_2 of endpoints c, d is contained in some $B' \in \mathcal{B}$.

Since $c \neq v$ it cannot be contained simultaneously in \mathcal{L}_1 and in \mathcal{L}_2 . Assume that $c \notin \mathcal{L}_1$. Then c is in the interior of a simple \mathcal{T} -bigon formed by \mathfrak{g}_1 and \mathcal{L}_1 . According to (3) this \mathcal{T} -bigon is contained in some $B'' \in \mathcal{B}$. The intersections $B_0 \cap B''$ and $B' \cap B''$ both contain non-trivial sub-arcs of \mathfrak{g}'_1 , therefore $B_0 = B'' = B'$. Thus the point d is in the set \mathcal{E} and it is strictly between c and v on \mathfrak{g}'_1 . This contradicts the choice of c .

We conclude that $c = v$. □

4 Asymptotically tree-graded metric spaces

4.1 Definition and properties

Let (X, dist) be a geodesic metric space and let $\mathcal{A} = \{A_i \mid i \in I\}$ be a collection of subsets of X . In every asymptotic cone $\text{Con}_\omega(X; e, d)$, we consider the collection \mathcal{A}_ω of limit subsets

$$\left\{ \lim_{\omega} (A_{i_n}) ; i = (i_n)^\omega \in I^\omega \text{ such that } \exists M_i \text{ with the property } \text{dist}(e_n, A_{i_n}) \leq_\omega M_i d_n \right\}.$$

Definition 4.1. The metric space X is *asymptotically tree-graded (ATG) with respect to \mathcal{A}* if every asymptotic cone $\text{Con}_\omega(X; e, d)$ is tree-graded with respect to \mathcal{A}_ω .

Following Convention 1.3, in the rest of the paper we shall assume that all ATG metric spaces are proper, that is no subset $A \in \mathcal{A}$ contains X in a tubular neighborhood of it.

The ATG property is meant as an extension of the property of (strong) relative hyperbolicity from groups to metric spaces. Theorem 1.4 emphasizes that it is the correct property to work with.

Remark 4.2. Let X be ATG with respect to $\mathcal{A} = \{A_i ; i \in I\}$.

- (1) It is easy to see that for every $\tau > 0$, the space X is ATG with respect to $\{\mathcal{N}_\tau(A_i) ; i \in I\}$.
- (2) More generally, let \mathcal{B} be a collection of subsets of X such that there exists a constant $K \geq 0$ and a bijection $\phi : \mathcal{A} \rightarrow \mathcal{B}$ verifying $\text{dist}_H(A, \phi(A)) \leq K$. Then X is ATG with respect to \mathcal{B} .

The notion of ATG metric space can also be defined by a list of geometric conditions, without involving asymptotic cones. First we introduce some notation and terminology.

Notation 4.3. Given \mathfrak{p} a quasi-geodesic and $r > 0$ we denote by $\check{\mathfrak{p}}_r$ the set $\mathfrak{p} \setminus \mathcal{N}_r(\{\mathfrak{p}_-, \mathfrak{p}_+\})$.

We say that a metric space P is a *geodesic (quasi-geodesic) k -gonal line* if it is a union of k geodesics (quasi-geodesics) $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ such that $(\mathfrak{q}_i)_+ = (\mathfrak{q}_{i+1})_-$ for $i = 1, \dots, k-1$. If moreover $(\mathfrak{q}_k)_+ = (\mathfrak{q}_1)_-$ then we say that P is a *geodesic (quasi-geodesic) k -gon*.

Let P be a quasi-geodesic polygon, with set of vertices \mathcal{V} . Points in $P \setminus \mathcal{V}$ are called *interior points of P* .

Notation 4.4. Given a vertex $x \in \mathcal{V}$ and \mathbf{q}, \mathbf{q}' the consecutive edges of P such that $x = \mathbf{q}_+ = \mathbf{q}'_-$, we denote the polygonal line $P \setminus (\mathbf{q} \cup \mathbf{q}')$ by $\mathcal{O}_x(P)$. When there is no possibility of confusion we simply denote it by \mathcal{O}_x .

Let $p \in P$. The *inscribed radius in p* with respect to P is either the distance from p to the set \mathcal{O}_p , if p is a vertex, or the distance from p to the set $P \setminus \mathbf{q}$ if p is an interior point contained in the edge \mathbf{q} (see Figure 2, taken from [DS05b]).

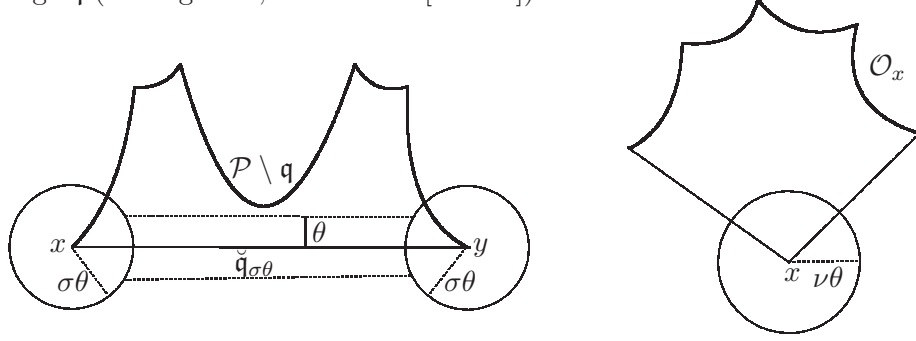


Figure 2: Properties (F_1) and (F_2) .

Definition 4.5 (fat polygon). Let $\theta > 0$, $\sigma \geq 1$ and $\nu \geq 4\sigma$. We call a k -gon P with quasi-geodesic edges (θ, σ, ν) -fat if the following properties hold:

(F_1) (**large inscribed radii in interior points, large comparison angles**) for every edge \mathbf{q} we have, with the notation 4.3, that

$$\text{dist}(\check{\mathbf{q}}_{\sigma\theta}, P \setminus \mathbf{q}) \geq \theta;$$

(F_2) (**large inscribed radii in vertices, large edges**) for every vertex x we have that

$$\text{dist}(x, \mathcal{O}_x) \geq \nu\theta.$$

When $\sigma = 2$ we say that P is (θ, ν) -fat.

Lemma 4.6. Let P be a polygon (θ, σ, ν) -fat for some $\theta > 0$, $\sigma \geq 1$ and $\nu \geq 4\sigma$. Then any two edges of P without a common vertex are at distance at least θ from each other.

Proof. Let \mathbf{q} and \mathbf{q}' be two edges without a common vertex. Assume that there exists a point $a \in \mathbf{q}$ such that $\text{dist}(a, \mathbf{q}') < \theta$. Property (F_1) implies that $a \in \mathcal{N}_{\sigma\theta}(\{x, y\})$, where x, y are the endpoints of \mathbf{q} . Property (F_2) implies that $\text{dist}(\{x, y\}, \mathbf{q}') \geq \nu\theta$. Therefore $\text{dist}(a, \mathbf{q}') \geq (\nu - \sigma)\theta \geq 3\sigma\theta \geq 3\theta$. This contradicts the assumption that $\text{dist}(a, \mathbf{q}') < \theta$. \square

The following lemma describes a situation in which given two consecutive edges of a geodesic polygon, any two points on each of these edges which are at distance at least 2θ from the common vertex are at distance at least θ from one another.

Lemma 4.7. Let P be a geodesic polygon with two consecutive edges $[x, y]$ and $[y, z]$ such that $\text{dist}(x, [y, z]) = \text{dist}(x, y)$. Then both the distance from $[x, y] \setminus B(y, 2\theta)$ to $[y, z]$, and the distance from $[y, z] \setminus B(y, 2\theta)$ to $[x, y]$ are at least θ .

Proof. The distance from $[x, y] \setminus B(y, 2\theta)$ to $[y, z]$ is 2θ because of the hypothesis that $\text{dist}(x, [y, z]) = \text{dist}(x, y)$.

Assume that there exists $p \in [y, z] \setminus B(y, 2\theta)$ and $p' \in [x, y]$ such that $\text{dist}(p, p') < \theta$. Then $\text{dist}(y, p') \geq \text{dist}(y, p) - \text{dist}(p, p') > \theta > \text{dist}(p, p')$. It follows that $\text{dist}(x, p) \leq \text{dist}(x, p') + \text{dist}(p', p) < \text{dist}(x, p') + \text{dist}(p', y) = \text{dist}(x, y)$. This contradicts the fact that $\text{dist}(x, [y, z]) = \text{dist}(x, y)$. \square

Theorem 4.8 ([DS05b], Theorem 4.1 and Remark 4.2, (3)). *Let (X, dist) be a geodesic metric space and let $\mathcal{A} = \{A_i \mid i \in I\}$ be a collection of subsets of X . The metric space X is asymptotically tree-graded with respect to \mathcal{A} if and only if the following properties are satisfied:*

- (α_1) *For every $\delta > 0$ the diameters of the intersections $\mathcal{N}_\delta(A_i) \cap \mathcal{N}_\delta(A_j)$ are uniformly bounded for all $i \neq j$.*
- (α_2) *There exists ε in $[0, \frac{1}{2})$ and $M > 0$ such that for every geodesic \mathbf{g} of length ℓ and every $A \in \mathcal{A}$ with $\mathbf{g}(0), \mathbf{g}(\ell) \in \mathcal{N}_{\varepsilon\ell}(A)$ we have that $\mathbf{g}([0, \ell]) \cap \mathcal{N}_M(A) \neq \emptyset$.*
- (α_3) *For every $k \geq 2$ there exist $\theta > 0$, $\nu \geq 8$ and $\chi > 0$ such that every k -gon P in X with geodesic edges which is (θ, ν) -fat satisfies $P \subset \mathcal{N}_\chi(A)$ for some $A \in \mathcal{A}$.*

Remarks 4.9 ([DS05b], Theorem 4.1 and Remark 4.2). (1) Property (α_2) from Theorem 4.8 is a slight modification of the similar property appearing in Theorem 4.1 in [DS05b]. Nevertheless it implies property (α_2^*) from [DS05b, Remark 4.2, (3)], which accounts for the accuracy of the modified statement.

- (2) As a necessary condition, (α_2) can be strengthened to “for every ε from $[0, \frac{1}{2})$ there exists $M > 0$ such that etc.”

Notation 4.10. We denote by diam_δ an uniform bound provided by property (α_1) for an arbitrary $\delta \geq 0$.

Remarks 4.11 (on the condition that pieces cover the space). (1) If in property (T_2) of Definition 3.1 of tree-graded spaces we allow for trivial geodesic triangles, that is if we ask that pieces cover a tree-graded space, then in Theorem 4.8 the following condition has to be added:

- (α_0) there exists $\tau \geq 0$ such that $X = \bigcup_{A \in \mathcal{A}} \mathcal{N}_\tau(A)$.

If (X, \mathcal{A}) satisfy only the conditions (α_1), (α_2), (α_3) but not (α_0) then it suffices to add some singletons to \mathcal{A} in order to ensure (α_0). Indeed, for some $\tau > 0$ consider in $X \setminus \bigcup_{A \in \mathcal{A}} \mathcal{N}_\tau(A)$ a maximal subset \wp with the property that $\text{dist}(p, p') \geq \tau$ for every $p, p' \in \wp$. The space X coincides with $\bigcup_{A \in \mathcal{A}} \mathcal{N}_\tau(A) \cup \bigcup_{p \in \wp} \mathcal{N}_\tau(\{p\})$. Properties (α_1) and (α_2) are obviously satisfied by singletons, whence X is ATG with respect to $\mathcal{A}' = \mathcal{A} \cup \{\{p\} \mid p \in \wp\}$; moreover \mathcal{A}' also satisfies (α_0).

- (2) Let \mathbb{H}^3 be the 3-dimensional real hyperbolic space and let $(Hbo_n)_{n \in \mathbb{N}}$ be a countable collection of pairwise disjoint open horoballs. The complementary set $X_0 = X \setminus \bigcup_{n \in \mathbb{N}} Hbo_n$ and the collection of boundary horospheres $\mathcal{A} = \{\partial Hbo_n \mid n \in \mathbb{N}\}$ is the typical example one has in mind when trying to define relative hyperbolicity for metric spaces. The pair (X_0, \mathcal{A}) does not in general satisfy (α_0), one has to add singletons to \mathcal{A} to ensure that property. In order to remove this inconvenient, we give up the condition of pieces covering the space in Definition 3.1 of tree-graded spaces.

Remark 4.12. If X is a metric space ATG with respect to \mathcal{A} , and a group G acts \mathcal{K} -transitively (in the sense of Definition 2.1, with $\mathcal{K} \geq 0$) by isometries on X , G permuting the subsets in \mathcal{A} , then property (α_0) is satisfied with $\tau = \mathcal{K}$.

It is for instance the case when X is itself a group and \mathcal{A} is the collection of left cosets of a family of subgroups.

4.2 Property (T_2) and polygons with limit edges

Property (α_3) in the definition of a metric space X ATG with respect to a collection \mathcal{A} is used to prove property (T_2) in an arbitrary asymptotic cone of X with respect to the collection of limit sets \mathcal{A}_ω . If X is such that any geodesic in an asymptotic cone of it is a limit geodesic (for instance if X is a CAT(0) metric space) then it suffices to have (α_3) for $k = 6$, that is:

(β_3) there exists $\theta > 0$, $\nu \geq 8$ and $\chi > 0$ such that any geodesic hexagon (θ, ν) -fat is contained in $\mathcal{N}_\chi(A)$, for some $A \in \mathcal{A}$.

This is due to the following general fact.

Proposition 4.13. *Let (X, dist) be a geodesic metric space and let $\theta > 0$ and $\nu \geq 8$ be two arbitrary constants. In any asymptotic cone $\text{Con}_\omega(X; e, d)$, any simple non-trivial triangle whose edges are limit geodesics is the limit set $\lim_\omega(H_n)$ of a sequence (H_n) of geodesic hexagons that are (θ, ν) -fat ω -almost surely.*

Proof. Consider a non-trivial simple geodesic triangle Δ in an asymptotic cone $\text{Con}_\omega(X; e, d)$, whose edges $[a, b]$, $[b, c]$ and $[c, a]$ appear as limit sets of sequences $[a_n, b'_n]$, $[b_n, c'_n]$ and $[c_n, a'_n]$ of geodesics in X . We have that $\text{dist}(a_n, a'_n)$, $\text{dist}(b_n, b'_n)$ and $\text{dist}(c_n, c'_n)$ are of order $o(d_n)$ ω -almost surely.

Let d_n^A be the maximum between $\text{dist}([a_n, b'_n], [a'_n, c_n])$ and $\nu\theta$. Note that $d_n^A > 0$ and that $d_n^A =_\omega o(d_n)$. Take a_n^1 to be the farthest from a_n point on $[a_n, b'_n]$ at distance d_n^A from $[a'_n, c_n]$. Consider then a_n^2 the farthest from a'_n point on $[a'_n, c_n]$ at distance d_n^A from a_n^1 . Obviously $\text{dist}(a_n^1, a_n^2) = d_n^A$.

The pairs of points (b_n^1, b_n^2) in $[b_n, c'_n] \times [b'_n, a_n]$, and respectively (c_n^1, c_n^2) in $[c_n, a'_n] \times [c'_n, b_n]$ are chosen similarly. Since the limit triangle Δ is simple, it follows that the sets $\{a_n, a'_n, a_n^1, a_n^2\}$, $\{b_n, b'_n, b_n^1, b_n^2\}$ and $\{c_n, c'_n, c_n^1, c_n^2\}$ have ω -almost surely diameters of order $o(d_n)$. Hence the sequence of geodesic hexagons H_n of vertices $a_n^1, b_n^2, b_n^1, c_n^2, c_n^1, a_n^2$ with edges $[a_n^1, b_n^2] \subset [a_n, b'_n]$, $[b_n^1, c_n^2] \subset [b_n, c'_n]$, $[c_n^1, a_n^2] \subset [c_n, a'_n]$, has the property that $\lim_\omega(H_n)$ is Δ . It remains to prove that H_n is ω -almost surely (θ, ν) -fat.

(F₁) The fact that the edge $[a_n^1, a_n^2]$ is at distance $O(d_n)$ from $[b_n^2, b_n^1] \cup [b_n^1, c_n^2] \cup [c_n^2, c_n^1]$ and Lemma 4.7 imply that $[a_n^1, a_n^2]$ satisfies property (F_1) .

In the same manner it can be shown that the edges $[b_n^1, b_n^2]$ and $[c_n^1, c_n^2]$ satisfy (F_1) .

The edge $[a_n^1, b_n^2]$ is at distance $O(d_n)$ from $[c_n^1, c_n^2]$. The choice of a_n^1 and of the pair (b_n^1, b_n^2) implies that $[a_n^1, b_n^2]$ is at distance at least $\nu\theta$ from $[b_n^1, c_n^2] \cup [c_n^1, a_n^2]$. Lemma 4.7 allows to conclude that $[a_n^1, b_n^2]$ satisfies (F_1) .

Similar arguments show that the edges $[b_n^1, c_n^2]$ and $[c_n^1, a_n^2]$ satisfy (F_1) .

(F₂) The distance from a_n^1 to $[a_n^2, c_n^1]$ is at least $\nu\theta$ by the choice of a_n^1 , while the distance to $[b_n^2, b_n^1] \cup [b_n^1, c_n^2] \cup [c_n^2, c_n^1]$ is $O(d_n)$. The same kind of argument shows that (F_2) is satisfied ω -almost surely by all the vertices of H_n . \square

In general not every geodesic in an asymptotic cone is a limit geodesic (see the example in the end of Section 2.2). Thus, in order to ensure property (T_2) in every asymptotic cone with

respect to the collection of limit sets \mathcal{A}_ω , in [DS05b] property (α_3) in full generality is used, together with the fact that limit sets are closed, and the following result.

Lemma 4.14 ([DS05b], Proposition 3.34). *Let Δ be an arbitrary simple geodesic triangle in $\text{Con}_\omega(X; e, d)$. For every $\varepsilon > 0$ sufficiently small there exists $k_0 = k_0(\varepsilon)$ and a simple geodesic triangle Δ_ε with the following properties:*

- (a) $\text{dist}_H(\Delta, \Delta_\varepsilon) \leq \varepsilon$;
- (b) Δ_ε contains the midpoints of the edges of Δ ;
- (c) for every $\theta > 0$ and $\nu \geq 8$ the triangle Δ_ε is the limit set $\lim_\omega(P_n^\varepsilon)$ of a sequence (P_n^ε) of geodesic k -gons in X , for some $k \leq k_0$, that are (θ, ν) -fat ω -almost surely.

Remark 4.15. If Δ is non-trivial then the set of midpoints of edges of Δ has cardinal 3, hence the triangles Δ_ε are also non-trivial.

In this section we prove that if in every asymptotic cone property (T_1) holds for the collection of limit sets \mathcal{A}_ω , then property (β_3) for the collection \mathcal{A} suffices to deduce (T_2) for \mathcal{A}_ω , again in every asymptotic cone. To this purpose, we define the following property in an asymptotic cone $\text{Con}_\omega(X; e, d)$:

(Π_k) every simple non-trivial k -gon with edges limit geodesics is contained in a subset from \mathcal{A}_ω .

Corollary 4.16. *Assume that in an asymptotic cone $\text{Con}_\omega(X; e, d)$ a collection \mathcal{A}_ω of closed subsets satisfies properties (T_1) and (Π_k) for every $k \in \mathbb{N}$, $k \geq 3$. Then \mathcal{A}_ω satisfies (T_2) .*

Proof. Consider a simple non-trivial geodesic triangle Δ in $\text{Con}_\omega(X; e, d)$. By Lemma 4.14 for every large enough $k \in \mathbb{N}$ there exists a simple non-trivial geodesic triangle Δ_k at Hausdorff distance at most $\frac{1}{k}$ from Δ , containing the midpoints of the edges of Δ , moreover $\Delta_k = \lim_\omega(P_n^{(k)})$, where $P_n^{(k)}$ is n - ω -almost surely a geodesic m -gon, $m = m(k)$. By property (Π_m) the triangle Δ_k is contained in some $A_k \in \mathcal{A}_\omega$. All A_k contain the midpoints of the edges of Δ . Property (T_1) implies that there exists $A \in \mathcal{A}_\omega$ such that $A_k = A$ for all k . All Δ_k are in A , Δ is the limit of Δ_k in the Hausdorff distance, and A is closed, therefore $\Delta \subset A$. \square

In view of Corollary 4.16 it suffices to prove that \mathcal{A}_ω satisfies (Π_k) for all $k \geq 3$ to deduce that \mathcal{A}_ω satisfies property (T_2) .

Obviously (Π_k) implies (Π_i) for every $i < k$. It turns out that with the additional assumption that (T_1) is satisfied, the converse implication also holds.

Lemma 4.17. *Assume that in an asymptotic cone $\text{Con}_\omega(X; e, d)$, the collection of subsets \mathcal{A}_ω satisfies the properties (T_1) and (Π_3) . Then \mathcal{A}_ω satisfies property (Π_k) for every $k \geq 3$.*

Proof. We prove property (Π_k) by induction on k . The cases $k = 2$ and $k = 3$ hold by hypothesis. Assume that the statement is true for every $k \leq m - 1$ and consider a simple non-trivial geodesic m -gon P in $\text{Con}_\omega(X; e, d)$, $m \geq 4$, with edges limit geodesics.

Let $[x, y]$ and $[y, z]$ be two consecutive edges of P , in clockwise order. Denote by \mathcal{L}_1 the union of the two edges $[x, y] \cup [y, z]$ of P , and by \mathcal{L} the union of the other $m - 2$ edges of P , in clockwise order.

Consider a limit geodesic \mathbf{g} joining x and z . If \mathbf{g} coincides with \mathcal{L}_1 or with \mathcal{L} then P is a simple geodesic polygon with at most $m - 1$ edges, all of them limit geodesics. By the inductive hypothesis P is contained in a subset A in \mathcal{A}_ω .

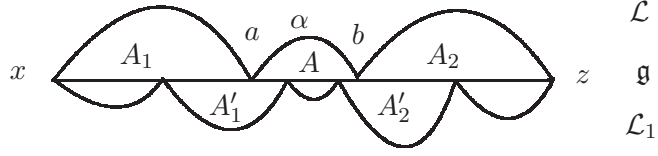


Figure 3: Step 1 in the proof of Lemma 4.17

Assume that \mathbf{g} does not coincide either with \mathcal{L}_1 or with \mathcal{L} .

STEP 1. We prove that $\mathbf{g} \cup \mathcal{L}$ is contained in some $A \in \mathcal{A}_\omega$.

Let $\alpha \in \mathcal{L} \setminus \mathbf{g}$. Lemma 3.7 implies that α is in the interior of a simple \mathcal{T} -bigon formed by \mathcal{L} and \mathbf{g} , of endpoints a and b , with a closer to x than b on \mathcal{L} . This \mathcal{T} -bigon is a geodesic polygon with at most $m - 1$ edges which are limit geodesics, therefore by the inductive hypothesis it is contained in a subset $A \in \mathcal{A}_\omega$.

If $a = x$ and $b = z$ then $\mathbf{g} \cup \mathcal{L}$ is a simple \mathcal{T} -bigon and it is contained in some $A \in \mathcal{A}_\omega$ by the inductive hypothesis. Assume therefore that $(a, b) \neq (x, z)$. Without loss of generality we may assume that $a \neq x$.

We apply Proposition 3.9 to \mathcal{L}_1 , $\mathbf{g}_1 = \mathbf{g}$, and $\mathbf{g}_2 = \mathcal{L}_2$ the sub-arc of \mathcal{L} in between x and α . Property (3) is satisfied by the hypothesis of the induction. It follows that the \mathcal{T} -bigon formed by \mathbf{g} and \mathcal{L} of endpoints x and a is contained in some $A_1 \in \mathcal{A}_\omega$.

The point a is in $\mathcal{L} \setminus \mathcal{L}_1$, hence it is in $\mathbf{g} \setminus \mathcal{L}_1$. By Lemma 3.7, a is in the interior of some simple \mathcal{T} -bigon formed by \mathcal{L}_1 and \mathbf{g} , and by (Π_3) this \mathcal{T} -bigon is in a subset $A'_1 \in \mathcal{A}_\omega$. Since $A'_1 \cap A$ and $A'_1 \cap A_1$ contain non-trivial sub-arcs of \mathbf{g} property (T_1) implies that $A = A'_1 = A_1$.

If moreover $b \neq z$, a similar argument gives that the \mathcal{T} -bigon formed by \mathbf{g} and \mathcal{L} of endpoints z and b is contained in A (see Figure 3).

We conclude that $\mathbf{g} \cup \mathcal{L}$ is contained in A .

STEP 2. We prove that \mathcal{L}_1 is also contained in A . Property (Π_3) implies that any non-trivial simple \mathcal{T} -bigon formed by \mathcal{L}_1 and \mathbf{g} is contained in a subset in \mathcal{A}_ω . We apply Lemma 3.8 to \mathcal{L}_1 and $\mathbf{g} \subset A$ and we conclude that $\mathcal{L}_1 \subset A$. Consequently A contains $\mathcal{L}_1 \cup \mathcal{L} = P$. \square

Corollary 4.18. *Assume that in an asymptotic cone $\text{Con}_\omega(X; e, d)$, the collection of closed subsets \mathcal{A}_ω satisfies properties (T_1) and (Π_3) . Then \mathcal{A}_ω satisfies property (T_2) .*

Corollary 4.19. *Let X be a geodesic metric space and \mathcal{A} a collection of subsets in X , such that (β_3) is satisfied and such that in any asymptotic cone $\text{Con}_\omega(X; e, d)$, the collection of limit subsets \mathcal{A}_ω satisfies property (T_1) . Then \mathcal{A}_ω satisfies property (T_2) .*

Note that the only thing missing in Corollary 4.19 to conclude that X is ATG with respect to \mathcal{A} is that \mathcal{A}_ω is composed of geodesic subsets.

Another useful consequence of Proposition 4.13 is the following.

Corollary 4.20. *Let (X, dist) be a geodesic metric space. Assume that for some $\theta > 0$ and $\nu \geq 8$ the set of (θ, ν) -fat geodesic hexagons is either empty or composed of hexagons of uniformly bounded diameter. Then X is hyperbolic.*

Proof. Proposition 4.13 implies that in any asymptotic cone of X any simple triangle with edges limit geodesics is trivial. This statement can be extended by induction to all polygons. Indeed, suppose that in any asymptotic cone of X for all $3 \leq k \leq m-1$ all simple k -gons with edges limit geodesics are trivial. Consider P a simple m -gon with edges limit geodesics in some $\text{Con}_\omega(X; e, d)$. Let $[x, y]$ and $[y, z]$ be two consecutive edges of P and let \mathbf{g} be a limit geodesic joining x and z . All simple \mathcal{T} -bigons formed by $[x, y] \cup [y, z]$ and \mathbf{g} must be trivial by the inductive hypothesis, thus $\mathbf{g} = [x, y] \cup [y, z]$. It follows that P is a simple $(m-1)$ -gon with edges limit geodesics, hence by the inductive hypothesis it is trivial.

Lemma 4.14 and Remark 4.15 imply that in any $\text{Con}_\omega(X; e, d)$ any simple geodesic triangle must be trivial. It follows that $\text{Con}_\omega(X; e, d)$ is a real tree, and since this holds for all asymptotic cones we conclude that X is hyperbolic ([Gro93, §2.4], see also [Dru02, §3]). \square

4.3 New definitions, useful for the rigidity of relatively hyperbolic groups

In this section new versions of the definition of an ATG metric space are stated and proved. They will play an important part in the proof of the quasi-isometric invariance of relative hyperbolicity.

Theorem 4.21. *In Theorem 4.8 the following modifications in the list of properties defining an asymptotically tree-graded metric space can be made:*

(M₁) property (α_3) can be replaced by property (β_3) ;

(M₂) property (α_2) can be either maintained or replaced by one of the following two properties:

- (β_2) there exists $\epsilon > 0$ and $M \geq 0$ such that for any geodesic \mathbf{g} of length ℓ and any $A \in \mathcal{A}$ satisfying $\mathbf{g}(0), \mathbf{g}(\ell) \in \mathcal{N}_{\epsilon\ell}(A)$, the middle third $\mathbf{g}([\frac{\ell}{3}, \frac{2\ell}{3}])$ is contained in $\mathcal{N}_M(A)$;
- (Ω) (**uniform quasi-convexity of pieces**) there exists $t > 0$ and $K_0 \geq 0$ such that for every $A \in \mathcal{A}$, $K \geq K_0$ and $x, y \in \mathcal{N}_K(A)$, every geodesic joining x and y in X is contained in $\mathcal{N}_{tK}(A)$.

Proof. Assume that X is ATG with respect to \mathcal{A} . The uniform quasi-convexity of pieces (Ω) is satisfied by [DS05b, Lemma 4.3]. Property (β_2) can be obtained for any $\epsilon < \frac{1}{6t}$, where t is the constant from (Ω), as follows. Consider a geodesic \mathbf{g} of length ℓ and $A \in \mathcal{A}$ as in (β_2) . We may assume that $\epsilon\ell \geq K_0$, otherwise \mathbf{g} would be contained in $\mathcal{N}_{\frac{K_0}{2\epsilon}}$. By (Ω) the geodesic \mathbf{g} is then contained in $\mathcal{N}_{t\epsilon\ell}(A)$. If $\theta = t\epsilon < \frac{1}{6}$ then by Theorem 4.8 and Remark 4.9 there exists $M = M(\theta)$ such that $\mathbf{g}([0, \frac{\ell}{3}])$ and $\mathbf{g}([\frac{2\ell}{3}, \ell])$ intersect $\mathcal{N}_M(A)$. Uniform convexity implies that $\mathbf{g}([\frac{\ell}{3}, \frac{2\ell}{3}])$ is contained in $\mathcal{N}_{tM'}(A)$, where $M' = \max(M, D_0)$.

Property (β_3) is a particular case of property (α_3) .

It remains to prove the converse statements: any of the triples of properties $(\alpha_1) \& (\alpha_2) \& (\beta_3)$, $(\alpha_1) \& (\beta_2) \& (\beta_3)$ or $(\alpha_1) \& (\Omega) \& (\beta_3)$ implies that X is ATG with respect to \mathcal{A} .

Lemma 4.22 ([DS05b], Lemma 4.3). *Properties (α_1) and (α_2) imply (Ω).*

Lemma 4.23. *Properties (α_1) and (β_2) imply (Ω) with K_0 equal to the constant M in (β_2) .*

Proof. Suppose by contradiction that for every $n \in \mathbb{N}^*$ there exists $A_n \in \mathcal{A}$, $K_n \geq M$ and $x_n, y_n \in \mathcal{N}_{K_n}(A_n)$ such that a geodesic $[x_n, y_n]$ is not contained in $\mathcal{N}_{nK_n}(A_n)$. For each $n \in \mathbb{N}^*$ we define D_n to be the infimum over the distances $\text{dist}(x_n, y_n)$ between pairs of points satisfying the properties above for some set in \mathcal{A} . In what follows we assume that we chose x_n, y_n at distance $\delta_n \leq D_n + 1$ of each other. Since $[x_n, y_n]$ is in $\mathcal{N}_{\delta_n/2}(\{x_n, y_n\}) \subset \mathcal{N}_{\delta_n/2+K_n}(A_n)$ it follows that $\frac{1}{2n-2}\delta_n \geq K_n$. In particular for n large enough $K_n < \epsilon\delta_n$, where $\epsilon > 0$ is the constant in (β_2) . It follows that the middle third $[a_n, b_n]$ of $[x_n, y_n]$ is contained in $\mathcal{N}_M(A_n)$. Since $K_n \geq M$, the fact that $[x_n, y_n] \not\subset \mathcal{N}_{nK_n}(A_n)$ implies that either $[x_n, a_n]$ or $[b_n, y_n]$ is not contained in $\mathcal{N}_{nK_n}(A_n)$. It follows that $D_n \leq \frac{\delta_n}{3} \leq \frac{D_n+1}{3}$, hence that the sequence (D_n) is uniformly bounded. This contradicts the fact that $D_n \geq (2n-2)M$. \square

Lemma 4.24. *Let P be a geodesic k -gon with two consecutive edges $[x, y]$ and $[y, z]$, such that $\text{dist}(x, [y, z]) = \text{dist}(x, y)$. If P is (θ, ν) -fat then the $(k+1)$ -gon P' obtained from P by adding as a vertex the point $v \in [x, y]$ with $\text{dist}(v, y) = \frac{\nu\theta}{2}$ is $(\theta, \frac{\nu}{2})$ -fat.*

Proof. Property (F_1) for P' follows easily from property (F_1) for P .

Property (F_2) holds for all the vertices different from x, v, y , by property (F_2) in P .

The polygonal line $\mathcal{O}_x(P') = \mathcal{O}_x(P) \cup [v, y]$ is in the $\frac{\nu\theta}{2}$ -tubular neighborhood of $\mathcal{O}_x(P)$, hence at distance at least $\frac{\nu\theta}{2}$ from x .

The polygonal line $\mathcal{O}_y(P')$ is equal to $\mathcal{O}_y(P) \cup [x, v]$. The line $\mathcal{O}_y(P)$ is at distance $\geq \nu\theta$ from y and $[x, v]$ is at distance $\frac{\nu\theta}{2}$ from y .

Finally, $\mathcal{O}_v(P') = \mathcal{O}_y(P) \cup [y, z]$. Since $\text{dist}(v, y) = \frac{\nu\theta}{2}$ it follows that $\text{dist}(v, \mathcal{O}_y) \geq \frac{\nu\theta}{2}$. If there exists $p \in [y, z]$ such that $\text{dist}(v, p) < \frac{\nu\theta}{2}$ then $\text{dist}(x, p) \leq \text{dist}(x, v) + \text{dist}(v, p) < \text{dist}(x, v) + \frac{\nu\theta}{2} = \text{dist}(x, y)$. This contradicts the hypothesis that $\text{dist}(x, [y, z]) = \text{dist}(x, y)$. \square

Lemma 4.25. *Properties (α_1) , (Ω) and (β_3) imply (α_2) for small enough $\varepsilon > 0$, and (β_2) .*

Proof. Assume that (Ω) and (β_3) are satisfied. Let $\mathbf{g} : [0, \ell] \rightarrow X$ be a geodesic with endpoints $x = \mathbf{g}(0)$ and $y = \mathbf{g}(\ell)$ contained in $\mathcal{N}_{\varepsilon\ell}(A)$ for some $A \in \mathcal{A}$. We shall prove that for a fixed positive constant D , the geodesic \mathbf{g} intersects $\mathcal{N}_D(A)$.

According to (Ω) , the geodesic \mathbf{g} is contained in $\mathcal{N}_{t\varepsilon\ell}(A)$.

Notation 4.26. We denote $t\varepsilon$ by ϵ and we assume in what follows that $\epsilon < \frac{1}{8}$. We denote by D the maximum between $tK_0 + 4\nu\theta$, χ and diam_δ (with the notation 4.10) for $\delta = \max(\chi, tK_0)$. Here t and K_0 are the constants appearing in (Ω) , while ν, θ, χ are the constants appearing in (β_3) .

Suppose by contradiction that \mathbf{g} does not intersect $\mathcal{N}_D(A)$. Note that since $\epsilon\ell \geq D$ it follows that $\ell > 8D$.

Consider x' and y' points in A such that $\text{dist}(x, x')$ and $\text{dist}(y, y')$ are at most $\varepsilon\ell$. By (Ω) , a geodesic \mathbf{g}' joining x' and y' is contained in $\mathcal{N}_{tK_0}(A)$.

Let $c \in \mathbf{g}$ and $c' \in \mathbf{g}'$ be two points such that $\text{dist}(c, c') = \text{dist}(\mathbf{g}, \mathbf{g}')$. Without loss of generality we may suppose that $\text{dist}(x, c) \geq \frac{\ell}{2}$. We may also suppose that $\text{dist}(x, x') = \text{dist}(x, \mathbf{g}')$. In order to transform the 4-gon of vertices x, x', c, c' into a fat polygon we make the following choices. Let x_1 be the point on \mathbf{g} between x and c which is farthest from x and at distance at most $2\nu\theta$ from $[x, x']$. Let x_2 be the farthest from x point on $[x, x']$ which is at distance $2\nu\theta$ from x_1 .

We prove in the sequel that the geodesic pentagon of vertices x_1, x_2, x', c', c is $(\theta, 2\nu)$ -fat. To simplify we shall denote its edges by $[v, w]$ if v, w are two consecutive vertices, keeping in mind that $[x_1, c] \subset \mathbf{g}$ and that $[x', c'] \subset \mathbf{g}'$.

(F₁) A point in $[c, c'] \setminus \mathcal{N}_{2\theta}(\{c, c'\})$ is at distance at least $(\frac{1}{2} - 2\epsilon)\ell$ from $[x, x']$, hence at distance at least $(\frac{1}{2} - 2\epsilon)\ell - 2\nu\theta$ of $[x_1, x_2]$. Since $\ell > 8D > 32\nu\theta$, it follows that $[c, c'] \setminus \mathcal{N}_{2\theta}(\{c, c'\})$ is at distance at least θ from $[x_1, x_2] \cup [x_2, x']$.

The choice of c, c' implies that all points in $[c, c'] \setminus \mathcal{N}_{2\theta}(\{c, c'\})$ are at distance at least 2θ from \mathbf{g} and from \mathbf{g}' .

The points in $[x_1, c] \setminus \mathcal{N}_{2\theta}(\{x_1, c\})$ are at distance at least $D - tK_0$ from \mathbf{g}' , and at distance at least $2\nu\theta$ from $[x_2, x']$. Lemma 4.7 allows to conclude that $[x_1, c]$ satisfies property (F₁).

The distance between $[x_1, x_2]$ and $[c, c']$ is at least $(\frac{1}{2} - 2\epsilon)\ell - 2\nu\theta$, and the one between $[x_1, x_2]$ and $[x', c']$ is at least $D - tK_0 - 2\nu\theta$. Thus, it suffices to verify that the distance between $[x_1, x_2] \setminus \mathcal{N}_{2\theta}(\{x_1, x_2\})$ and $[c, x_1] \cup [x_2, x']$ is at least θ . According to the choices of x_1, x_2 this distance is 2θ . This, and Lemma 4.7 also imply that $[x_2, x']$ satisfies (F₁).

The fact that the edge $[x', c']$ is at distance at least $D - tK_0 - 2\nu\theta$ from $\mathbf{g} \cup [x_1, x_2]$, together with Lemma 4.7, imply that $[x', c']$ satisfies (F₁).

(F₂) The vertex c' is at distance at least $D - tK_0 - 2\nu\theta$ from $[c, x_1] \cup [x_1, x_2]$ and at distance at least $(\frac{1}{2} - 2\epsilon)\ell$ from $[x_2, x']$.

The vertex c is at distance at least $D - tK_0$ from $[x', c']$ and at distance at least $(\frac{1}{2} - 2\epsilon)\ell - 2\nu\theta$ from $[x_1, x_2] \cup [x_2, x']$.

We have chosen x_1 at distance $2\nu\theta$ from $[x_2, x']$. The same vertex is at distance at least $D - tK_0$ from $[x', c']$, and at least $(\frac{1}{2} - 2\epsilon)\ell - 2\nu\theta$ from $[c, c']$.

Similarly, x_2 is at distance $2\nu\theta$ from $[x_1, c]$, at distance at least $(\frac{1}{2} - 2\epsilon)\ell - 2\nu\theta$ from $[c, c']$ and at least $D - tK_0 - 2\nu\theta$ from $[x', c']$.

The vertex x' is at distance at least $D - tK_0 - 2\nu\theta$ from $[c, x_1] \cup [x_1, x_2]$ and at least $(\frac{1}{2} - 2\epsilon)\ell$ from $[c, c']$.

The pentagon of vertices x_1, x_2, x', c', c is $(\theta, 2\nu)$ -fat. Lemma 4.24 and the fact that $\text{dist}(c, c') = \text{dist}(c, [c', x'])$ implies that by adding a vertex on $[c, c']$ this pentagon becomes a hexagon (θ, ν) -fat. Therefore by (β_3) it is contained in $\mathcal{N}_\chi(A')$ for some $A' \in \mathcal{A}$. In particular the edge $[x', c']$ is contained in $\mathcal{N}_\chi(A') \cap \mathcal{N}_{tK_0}(A)$. This edge has length at least $\frac{\ell}{4} > 2D$ and D is at least diam_δ for $\delta = \max(\chi, tK_0)$. It follows that $A = A'$ and that $D < \chi$, which is a contradiction.

We conclude that property (α_2) is satisfied for $\epsilon < \frac{1}{8t}$ and for D chosen above.

Property (β_2) is obtained as follows. If a geodesic $\mathbf{g} : [0, \ell] \rightarrow X$ joins two points in $\mathcal{N}_{\delta\ell}(A)$ then it is contained in $\mathcal{N}_{t\delta\ell}(A)$ by (\mathfrak{Q}) . If $\delta < \frac{\epsilon}{3t}$ then by (α_2) the sub-geodesics $g([0, \frac{\ell}{3}])$ and $g([\frac{2\ell}{3}, \ell])$ intersect $\mathcal{N}_M(A)$. Then by (\mathfrak{Q}) , $g([\frac{\ell}{3}, \frac{2\ell}{3}])$ is contained in $\mathcal{N}_{tM}(A)$. \square

In view of Lemmata 4.22, 4.23 and 4.25, in order to finish the proof of Theorem 4.21 it suffices to prove the following.

Lemma 4.27. *Assume that (X, dist) is a geodesic space and \mathcal{A} a collection of subsets, satisfying the properties (α_1) , (α_2) , (\mathfrak{Q}) and (β_3) . Then X is ATG with respect to \mathcal{A} .*

Proof. In an asymptotic cone $\text{Con}_\omega(X; e, d)$, the limit sets in \mathcal{A}_ω are closed. Property (\mathfrak{Q}) easily implies that all subsets in \mathcal{A}_ω are geodesic.

Property (T_1) for \mathcal{A}_ω is deduced from (α_1) and (α_2) as in [DS05b, Lemma 4.5]. Properties $(T_1) \& (\beta_3)$ imply (T_2) by Corollary 4.19. \square

Proposition 4.28. *For any $\eta > 0$ property (β_3) can be replaced by the following:*

(β_3^η) *there exists $\theta > 0$, $\nu \geq 8$ and $\chi > 0$ such that any geodesic hexagon (θ, ν) -fat of diameter at least η is contained in $\mathcal{N}_\chi(A)$, for some $A \in \mathcal{A}$.*

Proof. Indeed, as a sufficient condition (β_3^η) is used to prove (Π_3) in any asymptotic cone, by means of Proposition 4.13. Given the sequence of hexagons H_n in Proposition 4.13, H_n has ω -almost surely diameter of order $O(d_n)$. Property (β_3^η) suffices therefore to obtain property (Π_3) .

Property (β_3) is also used in Lemma 4.25 to prove (α_2) . It suffices to take in that proof the constant D larger than $tK_0 + \eta$ to obtain that the geodesic pentagon with vertices x_1, x_2, x', c', c has diameter at least $\text{dist}(c, c') \geq D - tK_0 > \eta$. That pentagon is $(\theta, 2\nu)$ -fat, hence by Lemma 4.24 it can be made into a hexagon (θ, ν) -fat of diameter larger than η ; therefore it is contained in $\mathcal{N}_\chi(A')$ for some $A' \in \mathcal{A}$. The rest of the argument is carried out similarly. \square

Corollary 4.29. *Let \mathcal{A}_{red}^η be the set of $A \in \mathcal{A}$ such that $\mathcal{N}_\chi(A)$ contains a (θ, ν) -fat geodesic hexagon of diameter at least η . Then the space X is ATG with respect to \mathcal{A}_{red}^η .*

Proof. Since $\mathcal{A}_{red}^\eta \subset \mathcal{A}$, properties (α_1) and (β_2) are satisfied. Property (β_3^η) is also satisfied by \mathcal{A}_{red}^η , hence by Proposition 4.28, X is ATG with respect to \mathcal{A}_{red}^η . \square

Corollary 4.30. *For every $\lambda > 0$ the space X is also ATG with respect to the subset \mathcal{A}_λ in \mathcal{A} composed of all the subsets of diameter at least λ in \mathcal{A} .*

Proof. Indeed $\mathcal{A}_\lambda \subset \mathcal{A}$ implies that properties (α_1) and (β_2) are still satisfied.

Let $\eta = \lambda + 2\chi$. Then $\mathcal{A}_{red}^\eta \subset \mathcal{A}_\lambda$, which implies that property (β_3^η) is satisfied by \mathcal{A}_λ . By Proposition 4.28, X is ATG with respect to \mathcal{A}_λ . \square

4.4 New definition, closer to the definition of hyperbolicity

In [DS05a] a version for groups of the following notion has been introduced.

Definition 4.31. Let X be a geodesic metric space and let \mathcal{A} be a collection of subsets of X . We say that X is $(*)$ -asymptotically tree-graded with respect to \mathcal{A} if for every $C \geq 0$ there exist two constants σ and δ such that every triangle xyz with $(1, C)$ -almost geodesic edges is in one of the following two cases:

- (C) there exists $a \in X$ such that $\overline{B}(a, \sigma)$ intersects each of the sides of the triangle;
- (P) there exists $A \in \mathcal{A}$ such that $\overline{\mathcal{N}}_\sigma(A)$ intersects each of the sides of the triangle, and the entrance (resp. exit) points x_1, y_1, z_1 (resp. y_2, z_2, x_2) of the sides $[x, y], [y, z], [z, x]$ in (from) $\overline{\mathcal{N}}_\sigma(A)$ satisfy

$$\text{dist}(x_1, x_2) < \delta, \text{dist}(y_1, y_2) < \delta, \text{dist}(z_1, z_2) < \delta.$$

See Figure 4, taken from [DS05b].

Remark 4.32. If X is a geodesic metric space in which for some constant $\sigma > 0$ every geodesic triangle satisfies property (C), then X is a hyperbolic space. Conversely, in a hyperbolic geodesic metric space for every $L \geq 1$ and $C \geq 0$ there exists $\sigma > 0$ such that every triangle with (L, C) -quasi-geodesic edges satisfies property (C).

Remarks 4.33. (1) If a metric space X is ATG with respect to a collection of subsets \mathcal{A} then X is $(*)$ -ATG with respect to \mathcal{A} , by [DS05b, Corollary 8.14 and Lemma 8.19].

Moreover, according to [DS05b, Corollary 8.14] if a geodesic triangle is in case (P) then for every $\sigma' \geq \sigma$ there exists δ' such that the pairs of entrance points in $\overline{\mathcal{N}}_{\sigma'}(A)$ are at distance at most δ' .

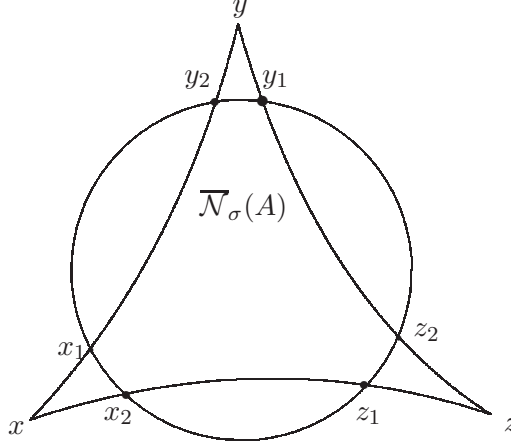


Figure 4: Case (P).

- (2) The notion of $(*)$ -ATG space is weaker than the one of ATG space. For instance if X is a geodesic hyperbolic space and if \mathcal{A} is any collection of subsets covering X , then X is $(*)$ -ATG with respect to \mathcal{A} , and the collection \mathcal{A} needs not satisfy properties (α_1) or (\mathfrak{Q}) , for instance.

It turns out nevertheless that one can formulate an equivalent definition of ATG metric spaces using the $(*)$ -property.

Theorem 4.34. *Let (X, dist) be a geodesic metric space and let \mathcal{A} be a collection of subsets of X . The metric space X is asymptotically tree-graded with respect to \mathcal{A} if and only if (X, \mathcal{A}) satisfy properties (α_1) and (α_2) , and moreover X is $(*)$ -ATG with respect to \mathcal{A} .*

Convention 4.35. In order to simplify some technical arguments of the equivalence we make the assumption that for all $C > 0$ the constant σ in the $(*)$ -property is larger than the constant M appearing in property (α_2) . By Remark 4.33, (1), if X is ATG then such a choice of σ is possible.

Proof. The direct implication has already been discussed, we now prove the converse statement. As in Section 4.3, from (α_1) and (α_2) can be deduced property (\mathfrak{Q}) . This property implies that in any asymptotic cone $\text{Con}_\omega(X; e, d)$ the collection \mathcal{A}_ω is composed of closed geodesic subsets.

Again (α_1) and (α_2) imply property (T_1) for \mathcal{A}_ω . According to Corollary 4.18, it remains to prove property (Π_3) .

Lemma 4.36. *Let (X, dist) be a geodesic metric space and let \mathcal{A} be a collection of subsets of X satisfying property (α_2) for some $\varepsilon \in [0, 1/2)$ and $M > 0$.*

Let $\mu \geq \nu \geq M$, let \mathfrak{g} be a geodesic and A a subset in \mathcal{A} such that \mathfrak{g} intersects $\overline{\mathcal{N}_\nu(A)}$. If e_μ and e_ν are the entrance points of \mathfrak{g} in $\overline{\mathcal{N}_\mu(A)}$ and respectively $\overline{\mathcal{N}_\nu(A)}$ then $\text{dist}(e_\mu, e_\nu) \leq \frac{\mu}{\varepsilon}$.

Proof. If $\varepsilon \text{dist}(e_\mu, e_\nu) > \mu$ then by (α_2) the sub-arc of \mathfrak{g} between e_μ and e_ν intersects $\mathcal{N}_M(A) \subset \mathcal{N}_\nu(A)$, which contradicts the definition of e_ν . \square

Lemma 4.37. *If (X, dist) is a geodesic metric space $(*)$ -ATG with respect to \mathcal{A} , and if moreover \mathcal{A} satisfies property (α_2) , then for every $C \geq 0$ there exist $\kappa \geq 0$ and $\lambda \geq 0$ such that the following holds. For any two geodesics $\mathfrak{g}, \mathfrak{g}'$ with $\mathfrak{g}_- = \mathfrak{g}'_-$ and $\text{dist}(\mathfrak{g}_+, \mathfrak{g}'_+) \leq C$, any point z on \mathfrak{g}' is either contained in $\overline{\mathcal{N}_\kappa(\mathfrak{g})}$ or it is contained in $\overline{\mathcal{N}_\kappa(A)}$ for some $A \in \mathcal{A}$ such that $\overline{\mathcal{N}_\kappa(A)}$ intersects \mathfrak{g} .*

Moreover in the latter case, if e, f and e', f' are the entrance and exit points from $\overline{\mathcal{N}}_\kappa(A)$ of \mathbf{g} and respectively \mathbf{g}' , then $\text{dist}(e, e'), \text{dist}(f, f') \leq \lambda$.

Proof. Let \mathbf{p} be the path $\mathbf{g} \sqcup [\mathbf{g}_+, \mathbf{g}'_+]$, where $[\mathbf{g}_+, \mathbf{g}'_+]$ is a geodesic segment joining \mathbf{g}_+ and \mathbf{g}'_+ . It is a $(1, 2C)$ -almost geodesic. Let σ and δ be the constants of property $(*)$ for $2C$, and let z be an arbitrary point on \mathbf{g}' , dividing \mathbf{g}' into two sub-arcs, \mathbf{g}_1 and \mathbf{g}_2 . The triangle Δ of edges $\mathbf{g}_1, \mathbf{g}_2$ and \mathbf{p} is either in case **(C)** or in case **(P)**.

If it is in case **(C)** then there exist $a_1 \in \mathbf{g}_1, a_2 \in \mathbf{g}_2$ and $b \in \mathbf{p}$ such that the set $\{a_1, a_2, b\}$ has diameter at most 2σ . The point z is on a geodesic joining a_1 and a_2 , hence it is at distance at most 3σ from b , thus it is contained in $\overline{\mathcal{N}}_{3\sigma+C}(\mathbf{g})$.

If Δ is in case **(P)** then there exists $A \in \mathcal{A}$ with $\overline{\mathcal{N}}_\sigma(A)$ intersecting $\mathbf{g}_1, \mathbf{g}_2$ and \mathbf{p} . Let x_1, z_1, z_2, y_1 and x_2, y_2 be the entrance and exit points from $\overline{\mathcal{N}}_\sigma(A)$ of $\mathbf{g}_1, \mathbf{g}_2$ and \mathbf{p} respectively. Then $\text{dist}(x_1, x_2), \text{dist}(y_1, y_2)$ and $\text{dist}(z_1, z_2)$ are all at most δ . Since z is on a geodesic joining z_1 and z_2 , $z \in \overline{\mathcal{N}}_{\sigma+\delta/2}(A)$. Note that $\overline{\mathcal{N}}_\sigma(A)$ intersects \mathbf{p} , therefore $\overline{\mathcal{N}}_{\sigma+C}(A)$ intersects \mathbf{g} .

Take $\kappa = \max(3\sigma + C, \sigma + \frac{\delta}{2}, \sigma + C)$.

The points x_1 and y_1 are the entrance and respectively the exit point of \mathbf{g}' from $\overline{\mathcal{N}}_\sigma(A)$. If we consider e' and f' the entrance and exit points of \mathbf{g}' from $\overline{\mathcal{N}}_\kappa(A)$, Lemma 4.36 implies that $\text{dist}(x_1, e')$ and $\text{dist}(y_1, f')$ are at most $\frac{\kappa}{\varepsilon}$. Hence $\text{dist}(e', x_2)$ and $\text{dist}(f', y_2)$ are at most $\frac{\kappa}{\varepsilon} + \delta$.

Let e and f be the entrance and exit points of \mathbf{g} from $\overline{\mathcal{N}}_\kappa(A)$. If either x_2 or both x_2 and y_2 are in \mathbf{g} then they are the entrance and respectively the exit point of \mathbf{g} from $\overline{\mathcal{N}}_\sigma(A)$. Lemma 4.36 implies that either $\text{dist}(x_2, e)$ or both $\text{dist}(x_2, e)$ and $\text{dist}(y_2, f)$ are at most $\frac{\kappa}{\varepsilon}$, hence that either $\text{dist}(e, e')$ or both $\text{dist}(e, e'), \text{dist}(f, f')$ are $O(1)$.

Assume that $y_2 \in [\mathbf{g}_+, \mathbf{g}'_+]$. Then \mathbf{g}_+ is in $\overline{\mathcal{N}}_\kappa(A)$, hence $\mathbf{g}_+ = f$. It follows that $\text{dist}(f, y_2) \leq C$ and that $\text{dist}(f, f') \leq C + \frac{\kappa}{\varepsilon} + \delta$.

Assume that $x_2 \in [\mathbf{g}_+, \mathbf{g}'_+]$. The point \mathbf{g}_+ is in $\overline{\mathcal{N}}_\kappa(A)$, and if $\varepsilon \text{dist}(e, \mathbf{g}_+) > \kappa$ then \mathbf{g} intersects $\mathcal{N}_M(A)$ between e and \mathbf{g}_+ . According to convention 4.35, $\mathcal{N}_M(A) \subset \mathcal{N}_\sigma(A)$, hence \mathbf{g} intersects $\mathcal{N}_\sigma(A)$ between e and \mathbf{g}_+ . This contradicts the fact that x_2 is the entrance point of \mathbf{p} into $\overline{\mathcal{N}}_\sigma(A)$. Thus $\text{dist}(e, \mathbf{g}_+) \leq \frac{\kappa}{\varepsilon}$ and $\text{dist}(e, x_2) \leq \frac{\kappa}{\varepsilon} + C$, whence $\text{dist}(e, e') \leq 2\frac{\kappa}{\varepsilon} + C + \delta$. \square

Lemma 4.38. *Let (X, dist) be a geodesic metric space $(*)$ -ATG with respect to a collection of subsets \mathcal{A} . Assume moreover that (X, \mathcal{A}) satisfy properties $(\alpha_1), (\alpha_2)$ and (\mathfrak{Q}) . Then there exist $\theta > 0, \nu \geq 8$ and $\chi > 0$ such that any geodesic quadrilateral which is (θ, ν) -fat is contained in $\mathcal{N}_\chi(A)$ for some $A \in \mathcal{A}$.*

Proof. Let P be a (θ, ν) -fat geodesic quadrilateral with vertices x, y, z, w in counterclockwise order. Let $[x, z]$ be a geodesic joining the opposite vertices x and z .

CASE 1. Assume that both geodesic triangles xyz and xzw are in case **(C)**. Then there exists $a_1 \in [x, y], a_2 \in [y, z]$ and $a_3 \in [x, z]$ such that the set $\{a_1, a_2, a_3\}$ has diameter at most 2σ . Likewise there exists $b_1 \in [z, w], b_2 \in [w, x]$ and $b_3 \in [z, x]$ such that $\{b_1, b_2, b_3\}$ has diameter at most 2σ . If $\theta > 2\sigma$ then $a_1, a_2 \in B(y, 2\theta)$ and $b_1, b_2 \in B(w, 2\theta)$.

Without loss of generality we may assume that $a_3 \in [x, b_3]$.

Notation 4.39. For $C = \max(2\sigma, \delta)$ let κ and λ be the constants given by Lemma 4.37.

Lemma 4.37 applied to $[x, b_2]$ and to $a_3 \in [x, b_3]$ implies that either $a_3 \in \overline{\mathcal{N}}_\kappa([x, b_2])$, or $a_3 \in \overline{\mathcal{N}}_\kappa(A)$ such that the entrance respectively exit point, a_4, a_5 , of $[x, b_3]$ from $\overline{\mathcal{N}}_\kappa(A)$ are at distance at most λ from $[x, b_2]$. If $\theta > 2\sigma + \kappa$ then by Lemma 4.6 the first case cannot occur.

In the second case we have that $\text{dist}(a_3, \{a_4, a_5\})$ is at least $\theta - 2\sigma - \lambda$. Lemma 4.37 applied to $[z, a_2]$ and to $a_5 \in [z, a_3]$ implies that either $a_5 \in \overline{\mathcal{N}}_\kappa([z, a_2])$ or that $a_5 \in \overline{\mathcal{N}}_\kappa(A')$ such that the

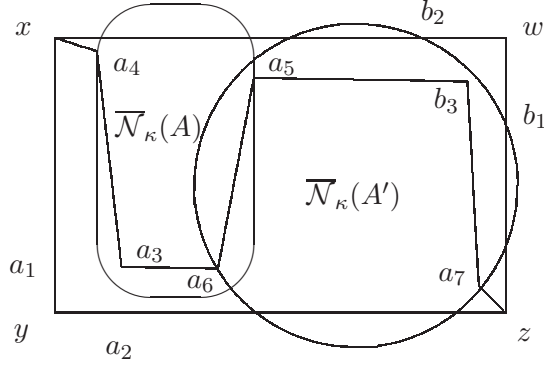


Figure 5: Case 1 in proof of Lemma 4.38.

entrance and the exit point a_6, a_7 , of $[a_3, z]$ from $\overline{\mathcal{N}}_\kappa(A')$ are at distance at most λ from $[z, a_2]$. The first case cannot occur if $\theta > \lambda + \kappa$. In the second case $\text{dist}(a_5, \{a_6, a_7\}) \geq \theta - 2\lambda$. The intersection $[a_3, a_5] \cap [a_6, a_7]$ has length at least $\theta - 2\sigma - 2\lambda$. By property (\mathfrak{Q}) this intersection is contained in $\mathcal{N}_{t\kappa+1}(A) \cap \mathcal{N}_{t\kappa+1}(A')$. If $\theta > 2\sigma + 2\lambda + \text{diam}_{t\kappa+1} + 1$ then $A = A'$.

The point a_4 is the entrance point of $[x, b_3]$ in $\overline{\mathcal{N}}_\kappa(A)$ while a_6 is the entrance point of $[a_3, z]$ in $\overline{\mathcal{N}}_\kappa(A)$. If $a_4 \in [a_3, b_3]$ then $a_4 = a_6$, and this point is at distance at most λ from both $[x, w]$ and $[y, z]$. If $\theta > 2\lambda$ then this cannot occur. Thus we may assume that $a_4 \in [x, a_3]$. Likewise we have that $a_7 \in [b_3, z]$ (see Figure 5).

We apply Lemma 4.37 to $[x, a_1]$ and to $a_4 \in [x, a_3]$. If we are in the second case of the conclusion then $a_4 \in \overline{\mathcal{N}}_\kappa(A'')$, and the entrance and exit point, a'_4, a''_4 , of $[x, a_3]$ from $\overline{\mathcal{N}}_\kappa(A'')$ are at distance at most λ from $[x, a_1]$. If $\text{dist}(a_4, a''_4) \geq \text{diam}_{t\kappa+1} + 1$ then $A'' = A$ and $a_4 = a'_4$. Thus, in all cases a_4 is at distance $O(1)$ from $[x, a_1]$. Recall that a_4 is at distance at most λ from $[x, w]$. It follows that if θ is large enough then $a_4 \in B(x, 2\theta + \lambda)$.

A similar argument gives that $a_7 \in B(z, 2\theta + \lambda)$.

We have thus that $\{x, z\} \subset \overline{\mathcal{N}}_{2\theta+\lambda+\kappa}(A)$. Also, since $\{a_3, b_3\} \subset [a_4, a_7] \subset \overline{\mathcal{N}}_{t\kappa}(A)$ it follows that $\{y, w\} \subset \overline{\mathcal{N}}_{2\theta+2\sigma+t\kappa}(A)$. By property (\mathfrak{Q}) , $P \subset \overline{\mathcal{N}}_\chi(A)$ where $\chi = t(2\theta + \lambda + 2\sigma + t\kappa)$.

CASE 2. Assume that the triangle xyz is in case **(P)** while xzw is in case **(C)**. Then there exists $A \in \mathcal{A}$ such that $\overline{\mathcal{N}}_\sigma(A)$ intersects all the edges of xyz . Moreover if x_2, y_1 are the entrance and exit point of $[x, y]$ in $\overline{\mathcal{N}}_\sigma(A)$, while y_2, z_1 and z_2, x_1 are the entrance and exit points of $[y, z]$ and respectively $[z, x]$ in $\overline{\mathcal{N}}_\sigma(A)$ then $\text{dist}(x_1, x_2), \text{dist}(y_1, y_2)$ and $\text{dist}(z_1, z_2)$ are at most δ .

Let also $b_1 \in [z, w]$, $b_2 \in [w, x]$ and $b_3 \in [x, z]$ be such that $\{b_1, b_2, b_3\}$ has diameter at most 2σ . If $\theta > 2\sigma$ then property (F_1) implies that $\{b_1, b_2\} \subset B(w, 2\theta)$.

CASE 2.A. Assume that $b_3 \in [x_1, z_2]$. Note that $\text{dist}(b_3, x_1) \geq \text{dist}(w, [x, y]) - 2\theta - 2\sigma - \delta \geq 6\theta - 2\sigma - \delta$. Same for $\text{dist}(b_3, z_2)$. Thus for θ large both $\text{dist}(b_3, x_1)$ and $\text{dist}(b_3, z_2)$ are large.

Lemma 4.37 applied to $x_1 \in [x, b_3]$ and to $[x, b'_2]$ implies that either $x_1 \in \overline{\mathcal{N}}_\kappa([x, b_2])$ or $x_1 \in \overline{\mathcal{N}}_\kappa(A_1)$ such that the entrance and exit points x'_1, x''_1 of $[x, b_3]$ from $\overline{\mathcal{N}}_\kappa(A_1)$ are at distance at most λ from $[x, b_2]$. In the latter case if $\text{dist}(x_1, x''_1) > \text{diam}_\tau$ where $\tau = t \max(\sigma, \kappa) + 1$ then $A_1 = A$ and $x_1 = x'_1$. Thus in all cases $\text{dist}(x_1, [x, w]) = O(1)$. For θ large enough it follows that $x_2 \in B(x, 2\theta)$, hence $x \in \overline{\mathcal{N}}_{2\theta+\sigma}(A)$. A similar argument gives that $z \in \overline{\mathcal{N}}_{2\theta+\sigma}(A)$.

If $\theta > \delta$ then $\text{dist}(y, y_1) < 2\theta$ and $y \in \overline{\mathcal{N}}_{2\theta+\sigma}(A)$.

points in $\overline{\mathcal{N}}_\sigma(A_1)$, (x_1, x_2) , (y_1, y_2) and (z_1, z_2) are all at respective distances less than δ . Likewise there exists A_2 in \mathcal{A} such that $\overline{\mathcal{N}}_\sigma(A_2)$ intersects all the edges of xzw , and the pairs of entrance points in $\overline{\mathcal{N}}_\sigma(A_2)$, (x'_1, x'_2) , (z'_1, z'_2) and (w_1, w_2) are all at distances less than δ (see Figure 7).

If $\theta > \delta$ then $y_1, y_2 \in B(y, 2\theta)$ and $w_1, w_2 \in B(w, 2\theta)$.

If $A_1 = A_2 = A$ then $x_1 = x'_1$, $z_2 = z'_1$, hence $\text{dist}(x'_1, x_2)$ and $\text{dist}(z_1, z'_2)$ are less than 2δ . If $\theta > 2\delta$ it follows that $x'_1, x_2 \in B(x, 2\theta)$ and that $z_1, z'_2 \in B(z, 2\theta)$. Thus $x, y, z, w \in \overline{\mathcal{N}}_{2\theta+\sigma}(A)$, which by (Ω) implies that $P \subset \mathcal{N}_\chi(A)$ for $\chi > t(2\theta + \sigma)$.

Assume that $A_1 \neq A_2$. Then $[x_1, z_2]$ and $[x'_2, z'_1]$ are either disjoint or they intersect in a sub-geodesic of length at most $\text{diam}_{t\sigma+1}$.

Let $\tau = t \max(\sigma, \kappa) + 1$. If either $[x_1, z_2]$ or $[x'_2, z'_1]$ is of length at most $\text{diam}_\tau + 1$ then either $\{x_2, x_1, z_1\}$ or $\{z'_1, z'_2, x'_1\}$ is of diameter at most $\text{diam}_\tau + 1 + 2\delta$. Therefore we find ourselves in Case 2.B, with the constant σ possibly larger. We can then finish the argument as in that case.

If both $[x_1, z_2]$ and $[x'_2, z'_1]$ have length larger than $\text{diam}_\tau + 1$ and their intersection is non-empty then either x'_1 and z_1 or x_2 and z'_2 are at distance at most $2\delta + \text{diam}_{t\sigma+1}$. If $\theta > 2\delta + \text{diam}_{t\sigma+1}$ then this is impossible. We may therefore assume that $[x_1, z_2]$ and $[x'_2, z'_1]$ do not intersect, and are both of length larger than $\text{diam}_\tau + 1$.

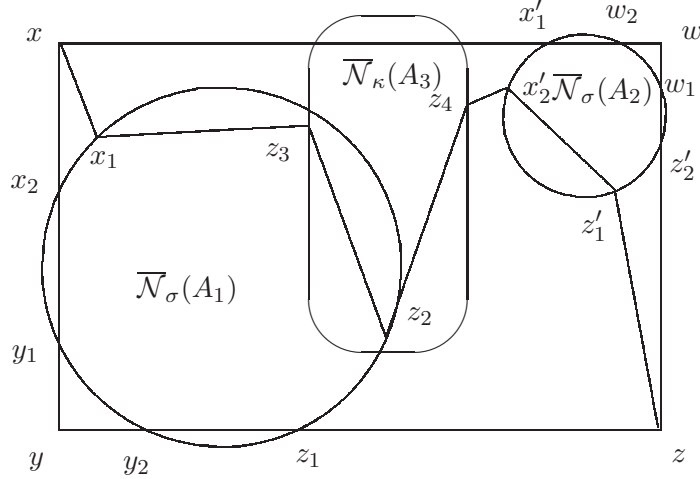


Figure 7: Case 3 in proof of Lemma 4.38.

Without loss of generality we may also assume that $[x_1, z_2] \subset [x, x'_2]$. Lemma 4.37 applied to the geodesic $[x, x'_1]$ and the point $z_2 \in [x, x'_2]$ implies that either $z_2 \in \overline{\mathcal{N}}_\kappa([x, w])$ or $z_2 \in \overline{\mathcal{N}}_\kappa(A_3)$, where $\overline{\mathcal{N}}_\kappa(A_3)$ intersects $[x, x'_1]$, and the entrance and exit points z_3, z_4 of $[x, x'_2]$ in $\overline{\mathcal{N}}_\kappa(A_3)$ are at distance at most λ from $[x, x'_1]$ (see Figure 7). If $\theta > \delta + \kappa$ then the first case cannot occur. In the second case we have that $\text{dist}(z_2, \{z_3, z_4\}) > \theta - \delta - \lambda$. In particular we may assume that $\text{dist}(z_2, z_3) > \text{diam}_\tau + 1$. We also have the assumption that $\text{dist}(x_1, z_2) > \text{diam}_\tau + 1$. Then $[x_1, z_2] \cap [z_3, z_2]$ has diameter $> \text{diam}_\tau$ and it is contained in $\overline{\mathcal{N}}_{t\sigma}(A_1) \cap \overline{\mathcal{N}}_{t\kappa}(A_3)$. Therefore $A_1 = A_3$. In particular z_4 , the exit point of $[x, x'_2]$ from $\overline{\mathcal{N}}_\kappa(A_1)$, and z_2 , the exit point of $[x, z]$ (therefore also of $[x, x'_2]$) from $\overline{\mathcal{N}}_\sigma(A_1)$ are at distance $O(1)$ by Lemma 4.36. It follows that z_1 and $[x, w]$ are at distance $O(1)$, and if θ is large enough this gives a contradiction. \square

Corollary 4.40. *Let (X, dist) and \mathcal{A} satisfy the hypotheses of Lemma 4.38. Then in any asymptotic cone of X property (Π_2) is satisfied by the collection of limit sets \mathcal{A}_ω .*

Proof. In an asymptotic cone $\text{Con}_\omega(X; e, d)$ consider a simple bigon of endpoints x, y whose edges are limit geodesics. Then there exist two sequences of geodesics $[x_n, y_n]$ and $[x'_n, y'_n]$ such that ω -almost surely $\text{dist}(x_n, x'_n)$ and $\text{dist}(y_n, y'_n)$ are of order $o(d_n)$, while $\text{dist}(x_n, y_n)$ and $\text{dist}(x'_n, y'_n)$ are of order $O(d_n)$. Let m_n and m'_n be the middlepoints of $[x_n, y_n]$ and respectively of $[x'_n, y'_n]$. Let δ_n be the maximum between $\text{dist}([x_n, m_n], [x'_n, m'_n])$ and $\nu\theta$, where θ and ν are the constants provided by Lemma 4.38. Then $\delta_n =_\omega o(d_n)$. Similarly, $\delta'_n = \max\{\text{dist}([m_n, y_n], [m'_n, y'_n]), \nu\theta\}$ satisfies $\delta'_n =_\omega o(d_n)$. Let x_n^1 be the farthest from x_n point on $[x_n, m_n]$ at distance δ_n from $[x'_n, m'_n]$, and let x_n^2 be the farthest from x'_n point on $[x'_n, m'_n]$ at distance δ_n from x_n^1 . We choose in a similar manner $y_n^1 \in [y_n, m_n]$ and $y_n^2 \in [y'_n, m'_n]$. Since the limit bigon is simple, it follows that the sets $\{x_n, x'_n, x_n^1, x_n^2\}$ and $\{y_n, y'_n, y_n^1, y_n^2\}$ have diameters of order $o(d_n)$ ω -almost surely.

We prove that any quadrilateral having as two opposite edges $[x_n^1, y_n^1] \subset [x_n, y_n]$ and $[x_n^2, y_n^2] \subset [x'_n, y'_n]$ is (θ, ν) -fat. This suffices to finish the argument, by Lemma 4.38.

(F₁) By construction $\text{dist}([x_n^1, y_n^1], [x_n^2, y_n^2]) \geq \nu\theta$, while the edges $[x_n^1, x_n^2]$ and $[y_n^1, y_n^2]$ are at distance $O(d_n)$ from each other. The rest of the property follows by Lemma 4.7.

Property (F₂) follows immediately from the fact that $\text{dist}([x_n^1, y_n^1], [x_n^2, y_n^2]) \geq \nu\theta$ and that $\text{dist}([x_n^1, x_n^2], [y_n^1, y_n^2]) = O(d_n)$. \square

The following statement ends the proof of Theorem 4.34.

Proposition 4.41. *Let (X, dist) be a geodesic metric space $(*)$ -ATG with respect to a collection of subsets \mathcal{A} . If (X, \mathcal{A}) moreover satisfy properties (α_1) and (α_2) , then in any asymptotic cone of X property (Π_3) is satisfied.*

Proof. Let $\text{Con}_\omega(X; e, d)$ be an arbitrary asymptotic cone of X and let \mathcal{A}_ω be the collection of limit sets of sequences from \mathcal{A} . Since $(\alpha_1) \& (\alpha_2) \Rightarrow (\Omega)$ it follows that the sets in \mathcal{A}_ω are geodesic. Also $(\alpha_1) \& (\alpha_2)$ imply that \mathcal{A}_ω satisfies (T_1) .

Let Δ be a non-trivial simple geodesic triangle in $\text{Con}_\omega(X; e, d)$, whose edges $[x, y]$, $[y, z]$ and $[z, x]$ appear as limits of sequences $[x_n, y'_n]$, $[y_n, z'_n]$ and $[z_n, x'_n]$ of geodesics in X . Then ω -almost surely $\text{dist}(x_n, x'_n)$, $\text{dist}(y_n, y'_n)$ and $\text{dist}(z_n, z'_n)$ are of order $o(d_n)$, while the lengths of $[x_n, y'_n]$, $[y_n, z'_n]$ and $[z_n, x'_n]$ are of order $O(d_n)$. Let T_n be a geodesic triangle with vertices x_n, y_n, z_n . We denote its edges by $[u, v]$, with $u, v \in \{x_n, y_n, z_n\}$. The three limit geodesics $\mathbf{g}^x = \lim_\omega([y_n, z_n])$, $\mathbf{g}^y = \lim_\omega([x_n, z_n])$ and $\mathbf{g}^z = \lim_\omega([x_n, y_n])$ compose the limit triangle $T = \lim_\omega(T_n)$.

CASE 1. Assume that ω -almost surely T_n is in case (C). Then there exists $a_n^1 \in [x_n, y_n]$, $a_n^2 \in [y_n, z_n]$ and $a_n^3 \in [z_n, x_n]$ such that the set $\{a_n^1, a_n^2, a_n^3\}$ has ω -almost surely diameter at most 2σ for some constant σ . It follows that $\lim_\omega(a_n^1) = \lim_\omega(a_n^2) = \lim_\omega(a_n^3) = a$. The point a is on the three edges of T .

Without loss of generality we may assume that $a \notin \{x, y\}$. The fact that $a \neq x$ implies that either $\mathbf{g}^z \neq [x, y]$ or $\mathbf{g}^y \neq [x, z]$. Property (Π_2) implies that we may apply Proposition 3.9 to $\mathcal{L}_1 = [x, y]$, $\mathcal{L}_2 = [x, z]$, $\mathbf{g}_1 = \mathbf{g}^z$, $\mathbf{g}_2 = \mathbf{g}^y$ and to the intersection point $a \in \mathbf{g}^z \cap \mathbf{g}^y$. We conclude that the \mathcal{T} -bigon formed by \mathbf{g}^z and \mathbf{g}^y of endpoints a, x is contained in a subset $A_x \in \mathcal{A}_\omega$.

Similarly we deduce that the \mathcal{T} -bigon formed by \mathbf{g}^z and \mathbf{g}^x of endpoints a, y is contained in a subset $A_y \in \mathcal{A}_\omega$.

If $a = z$ then $\mathbf{g}_x \subset A_y$, $\mathbf{g}_y \subset A_x$. Also $a \notin [x, y]$, which by Lemma 3.7 implies that a is contained in the interior of a simple \mathcal{T} -bigon formed by $[x, y]$ and \mathbf{g}^z . By property (Π_2) this \mathcal{T} -bigon is contained in some $A \in \mathcal{A}_\omega$. The intersections $A \cap A_x$ and $A \cap A_y$ contain non-trivial sub-arcs of \mathbf{g}^z therefore by (T_1) we conclude that $A = A_x = A_y$. The subset A contains also \mathbf{g}^z .

Property (Π_2) allows to apply Lemma 3.8 to the pairs of arcs $(\mathfrak{g}^x, [y, z])$, $(\mathfrak{g}^y, [x, z])$ and $(\mathfrak{g}^z, [x, y])$ and deduce that $\Delta = [x, y] \cup [y, z] \cup [z, x]$ is contained in A .

If $a \neq z$ then again by Proposition 3.9 the \mathcal{T} -bigon formed by \mathfrak{g}^x and \mathfrak{g}^y of endpoints a, z is contained in a subset $A_z \in \mathcal{A}_\omega$. Since a is not a vertex in Δ it is contained in at most one edge of Δ . Without loss of generality we assume that $a \notin [x, y] \cup [y, z]$.

The fact that $a \notin [x, y]$, Lemma 3.7, properties (Π_2) and (T_1) imply as above that $A_x = A_y$. Likewise from $a \notin [y, z]$ we deduce that $A_y = A_z$. Thus $A = A_x = A_y = A_z$ contains T . Property (Π_2) and Lemma 3.8 imply that Δ is also contained in A .

CASE 2. Assume that ω -almost surely T_n is in case **(P)**. Then there exist A_n in \mathcal{A} such that $\overline{\mathcal{N}}_\sigma(A_n)$ intersects all the edges of T_n . Moreover if (x_n^2, y_n^1) , (y_n^2, z_n^1) and (z_n^2, x_n^1) are the pairs of entrance and exit points from $\overline{\mathcal{N}}_\sigma(A)$ of $[x_n, y_n]$, $[y_n, z_n]$ and $[z_n, x_n]$ respectively, then $\text{dist}(x_n^1, x_n^2)$, $\text{dist}(y_n^1, y_n^2)$ and $\text{dist}(z_n^1, z_n^2)$ are less than δ .

Let $x' = \lim_\omega (x_n^1) = \lim_\omega (x_n^2)$, $y' = \lim_\omega (y_n^1) = \lim_\omega (y_n^2)$ and $z' = \lim_\omega (z_n^1) = \lim_\omega (z_n^2)$.

Assume that $\{x', y', z'\}$ has cardinal at most 2. Assume for instance that $x' = y'$. Then the point $a = x' = y'$ is in $\mathfrak{g}^x \cap \mathfrak{g}^y \cap \mathfrak{g}^z$. With the same argument as in Case 1 we deduce that both T and Δ are contained in some $A \in \mathcal{A}_\omega$.

Assume now that $\{x', y', z'\}$ has cardinal 3. The geodesic triangle T' of vertices x', y', z' and with edges contained in the edges of T is included in the piece $A = \lim_\omega (A_n)$.

Proposition 3.9 implies that the \mathcal{T} -bigon of endpoints x, x' formed by \mathfrak{g}^z and \mathfrak{g}^y is either trivial or contained in some $A_x \in \mathcal{A}_\omega$. Similarly, the \mathcal{T} -bigon of endpoints y, y' formed by \mathfrak{g}^z and \mathfrak{g}^x is either trivial or in some A_y , and the \mathcal{T} -bigon of endpoints z, z' formed by \mathfrak{g}^x and \mathfrak{g}^y is either trivial or in some A_z .

If $x' \neq x$ then x' cannot be contained both in $[x, y]$ and in $[x, z]$. Suppose that $x' \notin [x, y]$. Then x' is in the interior of a non-trivial simple \mathcal{T} -bigon formed by \mathfrak{g}^z and $[x, y]$. This \mathcal{T} -bigon is contained in some $B_x \in \mathcal{A}_\omega$ by (Π_2) , and its intersections with A_x and with A contain a non-trivial sub-arc of \mathfrak{g}^z . Hence $A_x = B_x = A$. Thus the \mathcal{T} -bigon of endpoints x, x' is contained in A .

In the same way we obtain that the \mathcal{T} -bigons of endpoints y, y' and z, z' are contained in A . Thus in all cases $T \subset A$, which by Lemma 3.8 implies that $\Delta \subset A$. \square

5 Quasi-isometric rigidity of relatively hyperbolic groups

In this section we prove the following.

Theorem 5.1. *Let (G, dist) be an infinite finitely generated group endowed with a word metric, which is asymptotically tree-graded with respect to a collection \mathcal{A} of subsets of G . Let \varkappa be the maximum between the constant M in property (β_2) and the constant χ in property (β_3) of (G, \mathcal{A}) .*

Then the group G is either hyperbolic or relatively hyperbolic with respect to a family of subgroups $\{H_1, \dots, H_m\}$, such that each H_i is contained in $\mathcal{N}_\varkappa(A_i)$ for some $A_i \in \mathcal{A}$.

Remark 5.2. If G is hyperbolic then it is hyperbolic relative to $H = \{1\}$. Still, in this case one cannot state that H is contained in some $\mathcal{N}_\varkappa(A)$ with $A \in \mathcal{A}$, because in the definition that we adopt of asymptotically tree-graded metric spaces the finite radius tubular neighborhoods of sets $A \in \mathcal{A}$ do not cover the whole space (see Remark 4.11).

Proof. The pair (G, \mathcal{A}) satisfies properties (α_1) , (β_2) and (β_3) .

By Corollary 4.20, if for $\theta > 0$ and $\nu \geq 8$ from (β_3) either there exists no (θ, ν) -fat geodesic hexagon in the Cayley graph of G , or the (θ, ν) -fat geodesic hexagons have uniformly bounded diameter, then G is hyperbolic.

Assume from now on that for every $\eta > 0$ there exists a (θ, ν) -fat geodesic hexagon of diameter at least η .

For \varkappa as in the theorem and diam_\varkappa given by property (α_1) of \mathcal{A} , consider the set

$$\Phi = \{P \text{ geodesic hexagon ; } P \text{ is } (\theta, \nu)\text{-fat , } \text{diameter}(P) \geq \text{diam}_\varkappa + 1\} .$$

Let $g \in G$. The metric space (G, dist) is asymptotically tree-graded with respect to the collection of subsets $g\mathcal{A} = \{gA ; A \in \mathcal{A}\}$, moreover the constants in the properties (α_1) , (β_2) and (β_3) are the same as for \mathcal{A} .

Let $P \in \Phi$. Then P is contained in $\mathcal{N}_\varkappa(gA)$ for some $A \in \mathcal{A}$. If P is also contained in $\mathcal{N}_\varkappa(gA')$ for $A' \in \mathcal{A}$ then $\mathcal{N}_\varkappa(A) \cap \mathcal{N}_\varkappa(A')$ has diameter at least the diameter of P , hence at least $\text{diam}_\varkappa + 1$, consequently $A = A'$. Thus P defines a map

$$\mathbf{A}_P : G \rightarrow \mathcal{A}, \mathbf{A}_P(g) = A \text{ such that } P \subset \mathcal{N}_\varkappa(gA) .$$

We may then define

$$\mathbf{A} : \Phi \rightarrow \text{Map}(G, \mathcal{A}), \mathbf{A}(P) = \mathbf{A}_P ,$$

where $\text{Map}(G, \mathcal{A})$ is the set of maps from G to \mathcal{A} . Consider the equivalence relation on Φ induced by \mathbf{A} , that is

$$P \sim P' \Leftrightarrow \mathbf{A}(P) = \mathbf{A}(P') \Leftrightarrow \forall g \in G, P \text{ and } P' \text{ are in the same } \mathcal{N}_\varkappa(gA) .$$

Let $[P]$ be the equivalence class of a hexagon P in Φ . To it we associate the set

$$B[P] = \bigcap_{g \in G} \mathcal{N}_\varkappa(g \mathbf{A}_P(g))$$

Proposition 5.3. *The metric space (G, dist) is ATG with respect to*

$$\mathcal{B} = \{B[P] ; [P] \in \Phi / \sim\} .$$

Proof. According to Proposition 4.28 it suffices to prove (α_1) , (β_2) and (β_3^η) for some $\eta > 0$. The proof relies on the simple remark that for every $r > 0$,

$$\mathcal{N}_r(B[P]) \subset \bigcap_{g \in G} \mathcal{N}_{r+\varkappa}(g \mathbf{A}_P(g)) .$$

(α_1) Let $[P] \neq [P']$, which is equivalent to the fact that there exists $g_0 \in G$ such that $P \subset \mathcal{N}_\varkappa(g_0A)$ and $P' \subset \mathcal{N}_\varkappa(g_0A')$ with $A \neq A'$. For every $\delta > 0$,

$$\mathcal{N}_\delta(B[P]) \cap \mathcal{N}_\delta(B[P']) \subset \mathcal{N}_{\delta+\varkappa}(g_0A) \cap \mathcal{N}_{\delta+\varkappa}(g_0A') = g_0 [\mathcal{N}_{\delta+\varkappa}(A) \cap \mathcal{N}_{\delta+\varkappa}(A')] .$$

Property (α_1) for \mathcal{A} implies that the diameter of $\mathcal{N}_\delta(B[P]) \cap \mathcal{N}_\delta(B[P'])$ is uniformly bounded.

(β_2) Let ϵ be the constant appearing in (β_2) for \mathcal{A} . Take $\epsilon' = \frac{\epsilon}{2}$ and take $M' = \frac{\epsilon+1}{\epsilon}\varkappa$. We prove that (β_2) holds for \mathcal{B} with the constants ϵ' and M' .

Let \mathbf{g} be a geodesic of length ℓ and let $[P] \in \Phi / \sim$ be such that $\mathbf{g}(0)$ and $\mathbf{g}(\ell)$ are in $\mathcal{N}_{\epsilon'\ell}(B[P])$. It follows that for every $g \in G$, $\mathbf{g}(0)$ and $\mathbf{g}(\ell)$ are in $\mathcal{N}_{\epsilon'\ell+\varkappa}(g \mathbf{A}_P(g))$.

If $\varkappa \geq \frac{\epsilon}{2}\ell \Leftrightarrow \ell \leq \frac{2\varkappa}{\epsilon}$ then $\mathbf{g} \subset \overline{\mathcal{N}}_{\frac{\varkappa}{\epsilon}}(\{\mathbf{g}(0), \mathbf{g}(\ell)\}) \subset \mathcal{N}_{\frac{\varkappa}{\epsilon}+\varkappa}(B[P]) = \mathcal{N}_{M'}(B[P])$.

Assume that $\varkappa < \frac{\varepsilon}{2}\ell$. Then for every $g \in G$ the geodesic $g^{-1}\mathfrak{g}$ of length ℓ has its end-points in $\mathcal{N}_{\varepsilon\ell}(\mathbf{A}_P(g))$. Property (β_2) implies that $g^{-1}\mathfrak{g}([\frac{\ell}{3}, \frac{2\ell}{3}])$ is contained in $\mathcal{N}_M(\mathbf{A}_P(g)) \subset \mathcal{N}_{\varkappa}(\mathbf{A}_P(g))$.

We have thus obtained that for every $g \in G$, $\mathfrak{g}([\frac{\ell}{3}, \frac{2\ell}{3}])$ is contained in $\mathcal{N}_{\varkappa}(g\mathbf{A}_P(g))$. It follows that $\mathfrak{g}([\frac{\ell}{3}, \frac{2\ell}{3}])$ is contained in $B[P]$.

Property (β_3'') holds for the constants θ and ν same as in (β_3) for \mathcal{A} , for the constant χ equal to 0, and $\eta = \text{diam}_{\varkappa} + 1$. Indeed every $P \in \Phi$ is contained in $B[P]$. \square

Lemma 5.4 (the group permutes the pieces). (1) If $P \sim P'$ and $g \in G$ then $gP \sim gP'$.

Consequently G acts on the left on Φ / \sim .

(2) For every $g \in G$ and $P \in \Phi$, $gB[P] = B[gP]$.

Proof. (1) The set $\mathbf{A}_P(\gamma)$ is defined by $P \subset \mathcal{N}_{\varkappa}(\gamma\mathbf{A}_P(\gamma))$. For every $g \in G$, $gP \subset \mathcal{N}_{\varkappa}(g\gamma\mathbf{A}_P(\gamma))$, hence $\mathbf{A}_{gP}(g\gamma) = \mathbf{A}_P(\gamma)$. From this can be deduced that $P \sim P' \Rightarrow gP \sim gP'$.

(2) The translate $gB[P] = \bigcap_{\gamma \in G} \mathcal{N}_{\varkappa}(g\gamma\mathbf{A}_P(\gamma))$ is equal to

$$\bigcap_{\gamma \in G} \mathcal{N}_{\varkappa}(g\gamma\mathbf{A}_{gP}(g\gamma)) = \bigcap_{\gamma' \in G} \mathcal{N}_{\varkappa}(\gamma'\mathbf{A}_{gP}(\gamma')) = B[gP].$$

\square

The following statement finishes the proof of Theorem 5.1.

Proposition 5.5 (equivariant ATG structure implies relative hyperbolicity). Consider a finitely generated group endowed with a word metric (G, dist) , which is ATG with respect to a collection of subsets \mathcal{B} , such that G permutes the subsets in \mathcal{B} .

Then G is either hyperbolic, or hyperbolic relative to a family of subgroups $\{H_1, \dots, H_m\}$ such that for each H_i there exists a unique $B_i \in \mathcal{B}$ satisfying $H_i \subset B_i \subset \mathcal{N}_{\mathcal{K}}(H_i)$, where \mathcal{K} is a constant depending only on (G, dist) and \mathcal{B} .

The proof is done in several steps.

Lemma 5.6. Finitely many subsets in \mathcal{B} contain 1.

Proof. By property (Ω) of \mathcal{B} there exists $\tau > 0$ such that for any x, y in some $B \in \mathcal{B}$ any geodesic $[x, y]$ is contained in $\mathcal{N}_{\tau}(B)$. Property (α_1) for \mathcal{B} implies that there exists D_{τ} such that for $B \neq B'$, $\mathcal{N}_{\tau}(B) \cap \mathcal{N}_{\tau}(B')$ has diameter at most D_{τ} .

Assume that $B \in \mathcal{B}$ contains 1 and has diameter at most $3D_{\tau}$. Then $B \subset \overline{B}(1, 3D_{\tau})$. As $\overline{B}(1, 3D_{\tau})$ is finite, only finitely many $B \in \mathcal{B}$ can be in this case.

Assume that B contains 1 and has diameter larger than $3D_{\tau}$. Then B contains some point x with $\text{dist}(1, x) > 3D_{\tau}$. The geodesic $[1, x]$ is contained in $\mathcal{N}_{\tau}(B)$ and it intersects the sphere around 1 of radius $2D_{\tau}$, $S(1, 2D_{\tau})$. We define a map from the set $\{B \in \mathcal{B} ; 1 \in B, \text{diam } B > 3D_{\tau}\}$ to the set of subsets of $S(1, 2D_{\tau})$, associating to each B the non-empty intersection $\mathcal{N}_{\tau}(B) \cap S(1, 2D_{\tau})$. By (α_1) and the choice of D_{τ} , two distinct subsets B, B' have disjoint images by the above map, in particular the map is injective. Since the set of subsets of $S(1, 2D_{\tau})$ is finite, so is the considered subset of \mathcal{B} . \square

Let $\mathcal{F} = \{B_1, B_2, \dots, B_k\}$ be the set of $B \in \mathcal{B}$ containing 1. For every $i \in \{1, 2, \dots, k\}$ let

$$\mathcal{I}_i = \{j \in \{1, 2, \dots, k\} \mid \exists g \in G \text{ such that } gB_i = B_j\}.$$

For every $j \in \mathcal{I}_i$ we fix $g_j \in G$ such that $g_j B_i = B_j$.

Notation 5.7. Define the constants $\mathcal{K}_i = \max_{j \in \mathcal{I}_i} \text{dist}(1, g_j)$ and $\mathcal{K} = \max_{1 \leq i \leq k} \mathcal{K}_i$.

Lemma 5.8. *For every $B \in \mathcal{B}$ the stabilizer $\text{Stab}(B) = \{g \in G \mid gB = B\}$ is a subgroup of G acting \mathcal{K} -transitively on B (in the sense of Definition 2.1).*

Proof. Let x and b be arbitrary points in B . Both subsets $b^{-1}B$ and $x^{-1}B$ contain 1 and are in \mathcal{B} . It follows that $b^{-1}B = B_i$ and $x^{-1}B = B_j$ for some $i, j \in \{1, 2, \dots, k\}$. Since $b^{-1}xB_j = B_i$ it follows that $j \in \mathcal{I}_i$ and that $B_j = g_j B_i$. The last equality can be re-written as $x^{-1}B = g_j b^{-1}B$ which implies that $xg_j b^{-1} \in \text{Stab}(B)$, hence that x is at distance at most $\text{dist}(1, g_j)$ from $\text{Stab}(B)b$. \square

Corollary 5.9. *For $i \in \{1, 2, \dots, k\}$, $\text{Stab}(B_i) \subset B_i \subset \mathcal{N}_{\mathcal{K}}(\text{Stab}(B_i))$.*

Let $\text{diam}_{2\mathcal{K}}$ be the uniform bound given by property (α_1) for (G, \mathcal{B}) and $\delta = 2\mathcal{K}$.

If all the subsets in \mathcal{B} have diameter at most $\text{diam}_{2\mathcal{K}} + 1$ then G is hyperbolic by Corollary 4.20. Thus, in what follows we may assume that \mathcal{B} contains subsets of diameter larger than $\text{diam}_{2\mathcal{K}} + 1$.

Denote by \mathcal{B}' the set of $B \in \mathcal{B}$ of diameter larger than $\text{diam}_{2\mathcal{K}} + 1$. Proposition 5.3 and Corollary 4.30 imply that G is ATG with respect to \mathcal{B}' . Obviously G also permutes the subsets in \mathcal{B}' .

Let $\mathcal{F}' = \mathcal{F} \cap \mathcal{B}'$. Let \mathcal{F}_0 be a subset of \mathcal{F}' such that for every $B \in \mathcal{F}'$, its orbit $G \cdot B$ intersects \mathcal{F}_0 in a unique element. Such a subset can be obtained for instance by considering one by one the elements B_i in \mathcal{F}' , and deleting from \mathcal{F}' all B_j with $j \in \mathcal{I}_i$, $j \neq i$.

It follows that for every $B \in \mathcal{B}'$, the orbit $G \cdot B$ intersects \mathcal{F}_0 in only one element.

Let $\bar{B}_1, \dots, \bar{B}_m$ be the elements of \mathcal{F}_0 .

Lemma 5.10. *For every $B \in \mathcal{B}'$ there exists a unique $j \in \{1, 2, \dots, m\}$ and a unique left coset $g\text{Stab}(\bar{B}_j)$ such that*

$$g\text{Stab}(\bar{B}_j) \subset B \subset \mathcal{N}_{\mathcal{K}}(g\text{Stab}(\bar{B}_j)) . \quad (3)$$

Proof. Existence. Let $g \in B$. Then $g^{-1}B \in \mathcal{B}'$ and $1 \in g^{-1}B$. Therefore $g^{-1}B = \bar{B}_j$ for some $j \in \{1, 2, \dots, m\}$. Corollary 5.9 implies the double inclusion (3).

Unicity. Assume that $g\text{Stab}(\bar{B}_j)$ and $g'\text{Stab}(\bar{B}_l)$ both satisfy (3), for $j, l \in \{1, 2, \dots, m\}$. Then

$$g\bar{B}_j \subset \mathcal{N}_{\mathcal{K}}(g\text{Stab}(\bar{B}_j)) \subset \mathcal{N}_{\mathcal{K}}(B) \subset \mathcal{N}_{2\mathcal{K}}(g'\text{Stab}(\bar{B}_l)) \subset \mathcal{N}_{2\mathcal{K}}(g'\bar{B}_l) .$$

Both $g\bar{B}_j$ and $g'\bar{B}_l$ are in \mathcal{B}' , in particular $g\bar{B}_j$ has diameter at least $\text{diam}_{2\mathcal{K}} + 1$. Property (α_1) implies that $g\bar{B}_j = g'\bar{B}_l$. According to the definition of \mathcal{F}_0 this can only happen if $j = l$. Then $g^{-1}g'$ is in $\text{Stab}(\bar{B}_j)$, and $g'\text{Stab}(\bar{B}_l)$ coincides with $g\text{Stab}(\bar{B}_j)$. \square

Lemma 5.11. *The group G is hyperbolic relative to $\{H_1, \dots, H_m\}$, where $H_j = \text{Stab}(\bar{B}_j)$.*

Proof. The fact that G is ATG with respect to \mathcal{B}' , Lemma 5.10 and Remark 4.2, (2), imply that G is ATG with respect to $\{gH_j ; g \in G/H_j, j \in \{1, 2, \dots, m\}\}$. In particular by (Ω) each H_j is quasi-convex in G , hence each H_j is finitely generated. Theorem 1.4 implies that G is hyperbolic relative to H_1, \dots, H_m .

If $G = H_j = \text{Stab}(\bar{B}_j)$ then Corollary 5.9 implies that $G = \bar{B}_j$. \square

A consequence of Theorem 5.1 is the following.

Theorem 5.12. *Let G be a group hyperbolic relative to a family of subgroups $\mathcal{H} = \{H_1, \dots, H_n\}$. If a group G' is (L, C) -quasi-isometric to G then G' is hyperbolic relative to $\mathcal{H}' = \{H'_1, \dots, H'_m\}$, where each H'_i can be embedded (λ, κ) -quasi-isometrically in H_j for some $j = j(i) \in \{1, 2, \dots, n\}$, where (λ, κ) depend on (L, C) and on (G, \mathcal{H}) .*

Proof. If the group G is finite then the group G' is also finite. We assume henceforth that both groups are infinite.

Let q be an (L, C) -quasi-isometry from G to G' , and let \bar{q} be its quasi-converse, such that $\text{dist}(q \circ \bar{q}, \text{id}_{G'}) \leq D$ and $\text{dist}(\bar{q} \circ q, \text{id}_G) \leq D$, where $D = D(L, C)$. By Theorem 1.4, G is ATG with respect to the collection of left cosets $\mathcal{A} = \{gH_i ; g \in G/H_i, i \in \{1, 2, \dots, n\}\}$. Theorem 5.1 in [DS05b] implies that G' is ATG with respect to $q(\mathcal{A}) = \{q(A) ; A \in \mathcal{A}\}$. Moreover all constants appearing in the properties (α_i) , $i = 1, 2, 3$, (β_j) , $j = 2, 3$, and (\mathfrak{Q}) for $(G', q(\mathcal{A}))$ can be expressed as functions of (L, C) and of the constants in the similar properties for (G, \mathcal{A}) .

Theorem 5.1 implies that G' is either hyperbolic or relatively hyperbolic with respect to a family of subgroups $\{H'_1, \dots, H'_m\}$; moreover each H'_i is contained in $\mathcal{N}_\varkappa(q(A_i))$ for some $A_i \in \mathcal{A}$, where \varkappa is a constant depending on (L, C) , on the constant M in (β_2) for (G, \mathcal{A}) , and on the constant χ in (β_3) for (G, \mathcal{A}) .

Let $\pi_1 : \mathcal{N}_\varkappa(q(A_i)) \rightarrow q(A_i)$ be a map such that $\text{dist}(x, \pi_1(x)) \leq \varkappa$. Then π_1 is a $(1, 2\varkappa)$ -quasi-isometric embedding. Let $\pi_2 : \mathcal{N}_D(A_i) \rightarrow A_i$ be a $(1, 2D)$ -quasi-isometric embedding constructed similarly. The restriction to H'_i of $\pi_2 \circ \bar{q} \circ \pi_1$ is a (λ, κ) -quasi-isometric embedding of H'_i into $A_i = gH_j$, for some $j \in \{1, 2, \dots, n\}$, with (λ, κ) depending on (L, C) , \varkappa and D .

If G' is hyperbolic then G' is relatively hyperbolic with respect to $\{1\} \neq \{G'\}$ and all the statements in the theorem hold.

If $G' = H'_i$ then $G' = \mathcal{N}_\varkappa(q(A_i))$, which implies that $G \subset \mathcal{N}_C(\bar{q}(G')) \subset \mathcal{N}_{L\varkappa+2C+D}(A_i)$. By Theorem 1.4, this contradicts the fact that G is (properly) hyperbolic relative to \mathcal{H} . \square

References

- [AB05] E. Alibegovic and M. Bestvina, *Limit groups are CAT(0)*, Preprint ARXIV:MATH.GR/0410198, 2005.
- [Beh05] J. Behrstock, *Asymptotic geometry of the mapping class group and Teichmüller space*, Preprint ARXIV:MATH.GT/0502367, 2005.
- [BDM05] J. Behrstock, C. Druţu, and L. Mosher, *Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity*, Preprint ARXIV:MATH.GT/0512592, 2005.
- [BF01] J. Brock and B. Farb, *Curvature and rank of Teichmüller space*, Preprint ARXIV:MATH.GT/0109045, 2001.
- [Bou65] Nicolas Bourbaki, *Topologie générale*, Hermann, Paris, 1965.
- [Bow97] B. Bowditch, *Relatively hyperbolic groups*, Preprint, University of Southampton, <http://www.maths.soton.ac.uk/pure/preprints.phtml>, 1997.
- [Dah03a] F. Dahmani, *Combination of convergence groups*, Geometry and Topology **7** (2003), 933–963.
- [Dah03b] F. Dahmani, *Les groupes relativement hyperboliques et leurs bords*, Ph.D. thesis, University of Strasbourg, 2003.

- [dDW84] L. Van den Dries and A. Wilkie, *On Gromov's theorem concerning groups of polynomial growth and elementary logic*, J. Algebra **89** (1984), 349–374.
- [Dru02] C. Druţu, *Quasi-isometry invariants and asymptotic cones*, Int. J. Algebra Comput. **12** (2002), 99–135.
- [Dru04] C. Druţu, *Quasi-isometry rigidity of groups*, Preprint, 2004.
- [DS05a] C. Druţu and M. Sapir, *Relatively hyperbolic groups with Rapid Decay property*, Int. Math. Res. Notices **19** (2005), 1181–1194.
- [DS05b] C. Druţu and M. Sapir, *Tree-graded spaces and asymptotic cones of groups*, Topology **44** (2005), 959–1058. With an appendix by D. Osin and M. Sapir.
- [Dyu00] A. Dyubina, *Instability of the virtual solvability and the property of being virtually torsion-free for quasi-isometric groups*, Internat. Math. Res. Notices **21** (2000), 1097–1101.
- [EFW05] A. Eskin, D. Fisher and K. Whyte, *Quasi-isometries and rigidity of solvable groups*, Preprint ARXIV:MATH.GR/0511647, 2005.
- [Far97] B. Farb, *The quasi-isometry classification of lattices in semisimple Lie groups*, Math. Res. Letters **4** (1997), no. 5, 705–717.
- [Far98] B. Farb, *Relatively hyperbolic groups*, Geom. Funct. Anal. **8** (1998), no. 5, 810–840.
- [FM98] B. Farb and L. Mosher, *A rigidity theorem for the solvable Baumslag-Solitar groups*, Invent. Math. **131** (1998), no. 2, 419–451, Appendix by D. Cooper.
- [FM99] B. Farb and L. Mosher, *Quasi-isometric rigidity for the solvable Baumslag-Solitar groups, II*, Invent. Math. **137** (1999), no. 3, 613–649.
- [GdlH90] E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, no. 83, Birkhauser, 1990.
- [Gro81] M. Gromov, *Groups of polynomial growth and expanding maps*, IHES Sci. Publ. Math. **53** (1981), 53–73.
- [Gro87] M. Gromov, *Hyperbolic groups*, Essays in group theory (S. Gersten, ed.), MSRI Publications, vol. 8, Springer, 1987.
- [Gro93] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, Vol. 2 (Sussex, 1991) (G. Niblo and M. Roller, eds.), LMS Lecture Notes, vol. 182, Cambridge Univ. Press, 1993, pp. 1–295.
- [Hru04] C. Ch. Hruska, *Nonpositively curved 2-complexes with isolated flats*, Geometry and Topology **8** (2004), 205–275.
- [HK05] C. Ch. Hruska and B. Kleiner, *Hadamard spaces with isolated flats*, Geometry and Topology **9** (2005), 1501–1538.
- [KL95] M. Kapovich and B. Leeb, *On asymptotic cones and quasi-isometry classes of fundamental groups of nonpositively curved manifolds*, Geom. Funct. Anal. **5** (1995), no. 3, 582–603.

- [KL97] M. Kapovich and B. Leeb, *Quasi-isometries preserve the geometric decomposition of Haken manifolds*, Invent. Math. **128** (1997), no. 2, 393–416.
- [KL98] M. Kapovich and B. Leeb, *3-manifold groups and nonpositive curvature*, Geom. Funct. Anal. **8** (1998), no. 5, 841–852.
- [KKL98] M. Kapovich, B. Kleiner and B. Leeb, *Quasi-isometries and the de Rham decomposition*, Topology **37** (1998), no. 6, 1193–1211.
- [KS04] I. Kapovich and P. Schupp, *Relative hyperbolicity and Artin groups*, Geometriae Dedicata **107** (2004), 153–167.
- [OOS06] A.Yu. Ol’shanskii, D.V. Osin and M.V. Sapir, *Cut points in asymptotic cones of groups*, Preprint, 2006.
- [Osi04] D.V. Osin, *Small cancellations over relatively hyperbolic groups and embedding theorems*, Preprint arXiv:math.GR/0411039, 2004.
- [Osi06] D.V. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. AMS **179** (2006), no. 843, vi+100pp.
- [PW02] P. Papasoglu and K. Whyte, *Quasi-isometries between groups with infinitely many ends*, Comment. Math. Helv. **77** (2002), no. 1, 133–144.
- [Sch96] R. Schwartz, *The quasi-isometry classification of rank one lattices*, IHES Sci. Publ. Math. **82** (1996), 133–168.
- [Val04] A. Valette, *Nouvelles approches de la propriété (T) de Kazhdan*, Astérisque **294** (2004), 97–124.
- [Yam04] A. Yaman, *A topological characterisation of relatively hyperbolic groups*, J. Reine Angew. Math. (Crelle’s Journal) **566** (2004), 41–89.

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